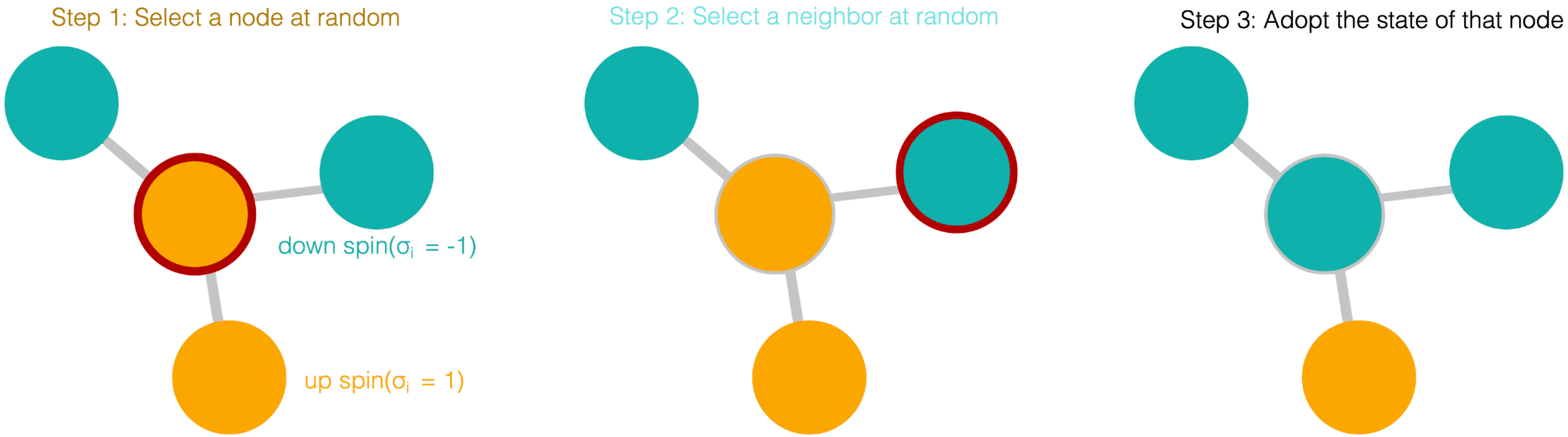
Groups Groups Groups

Social interactions are not pairwise - groups matter!

How do higher-order interactions affect the development of consensus?

Voter Models

Linear Voter Model



Voter Model Steps

1. A random node i with state $\sigma_i \in \{-1, 1\}$, is selected
2. The selected node adopts the spin σ_j of a randomly selected neighbor $j \in \mathcal{N}_i$
3. Process is repeated until consensus is reached.
4. Transition rate for a node $\dot{\sigma}_i \propto \text{fraction of disagreeing neighbors}$

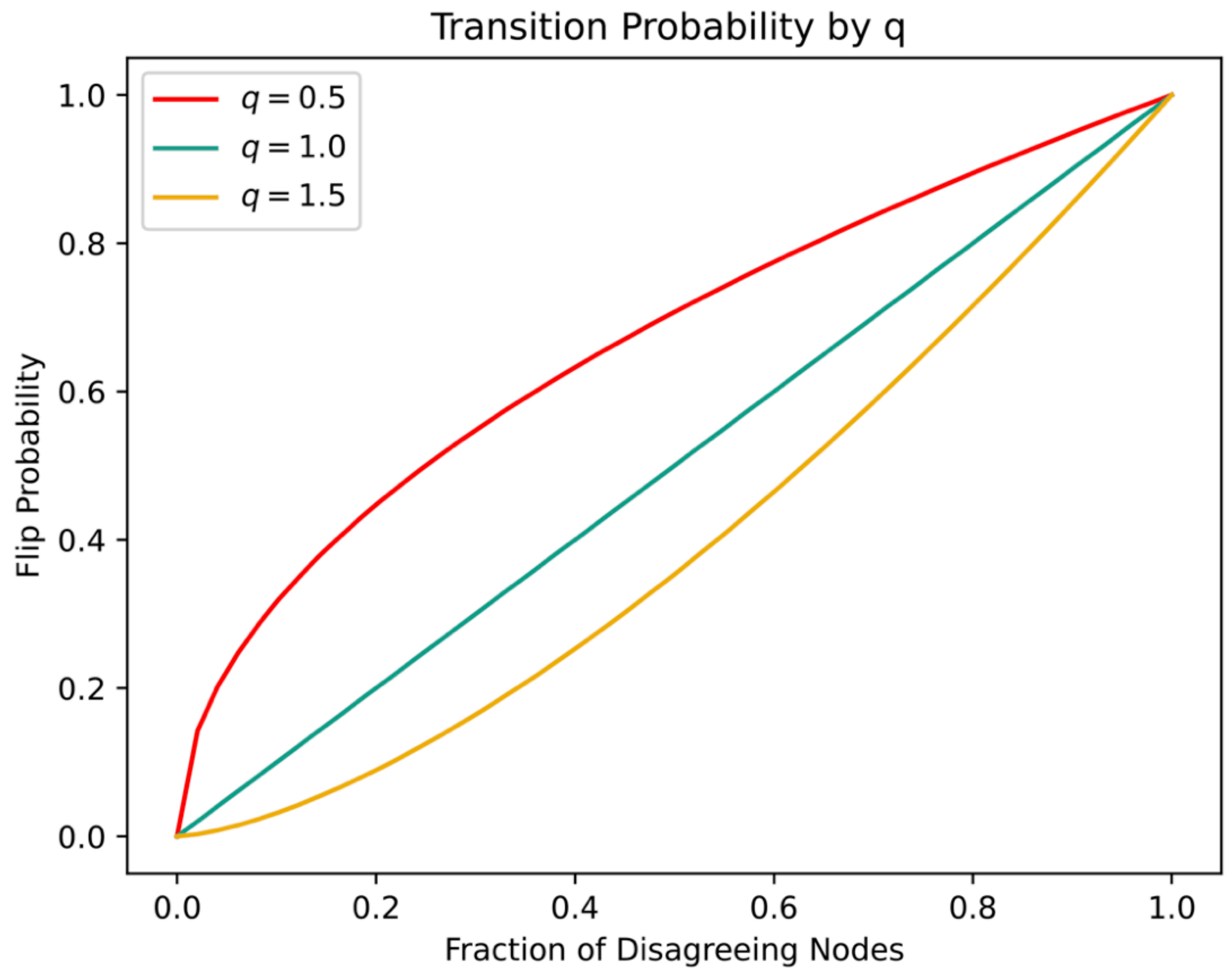
Voter Models
Non-Linear

Q Voter Model Steps[a]

1. A random node σ_i selects q of its neighbors. If all of its neighbors have the same spin, σ_i adopts that spin
2. Transition rate for a node
   $\dot{\sigma}_i \propto \text{fraction of disagreeing neighbors}^q$

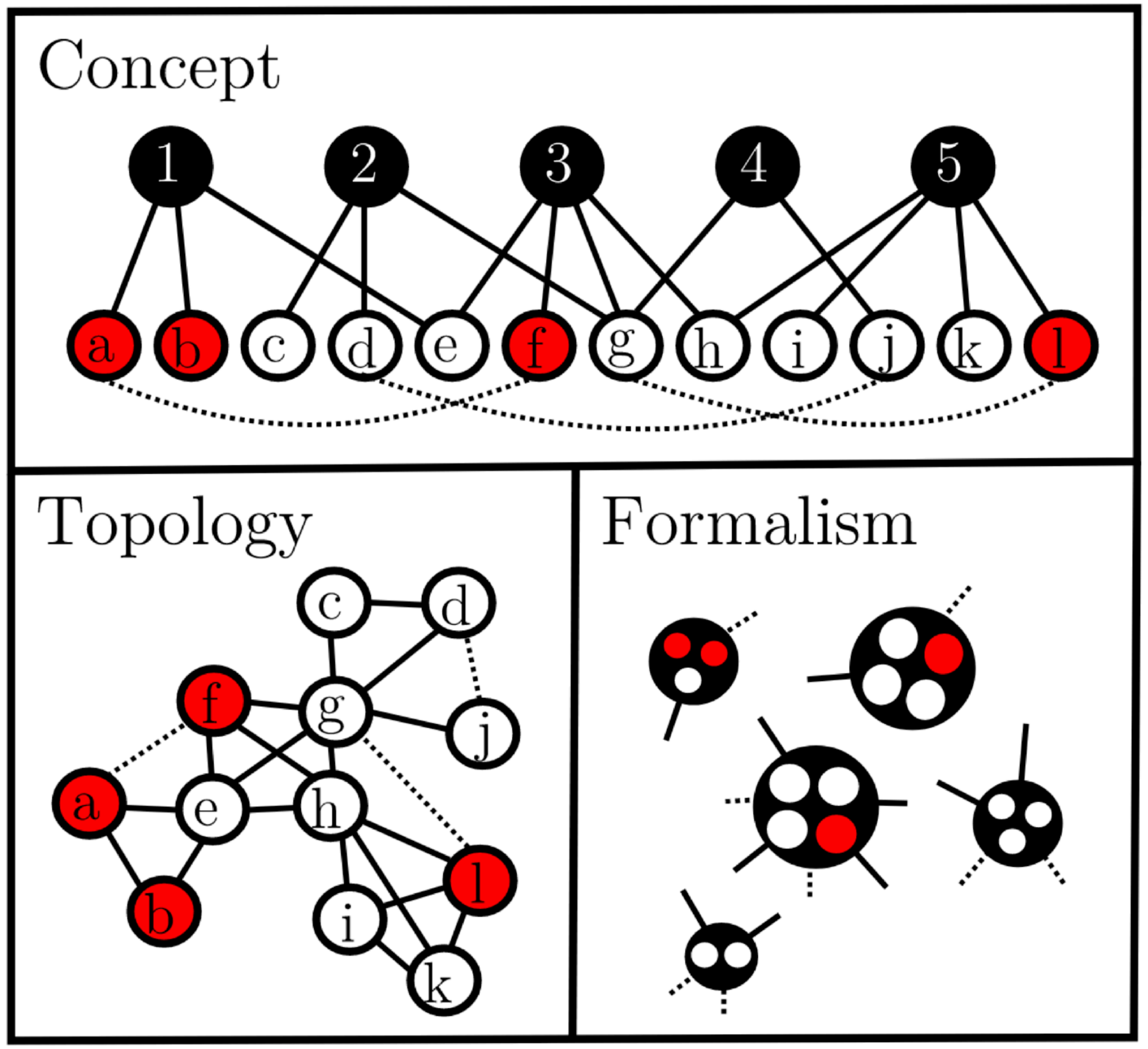
What does q do?

- q controls the conformity bias of the model.
- if $q > 1$: conformist nodes nodes, if $q < 1$, we get the hipster nodes nodes.



[a] Castellano et al., 2009.

- Each node belongs to a set of cliques. Nodes interact with other nodes in the same clique.
- Higher-order network topology generated by the model proposed by Newman[a].



Description

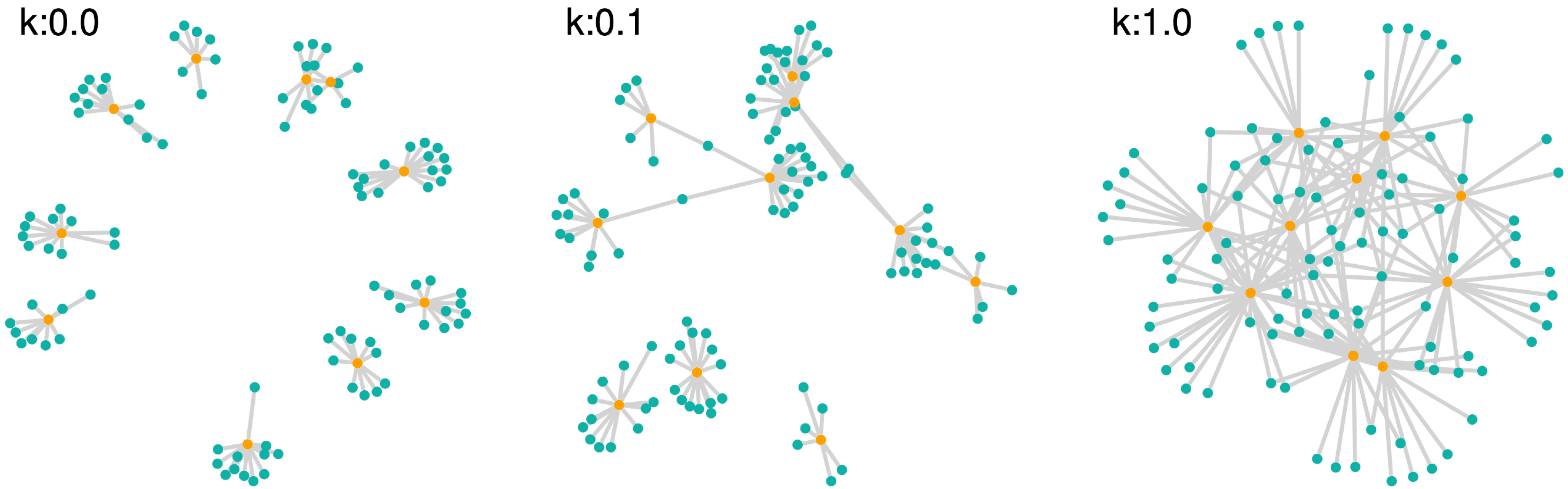
The model is parameterized by two distributions

1. N the number of nodes
2. M the number of cliques
3. $\{p_n\}$ the distribution of nodes per clique
4. $\{g_m\}$ the distribution of cliques per node

[a] Newman, 2003.

Clique Coupling

$\langle k_{ex} \rangle$ determines the coupling between groups

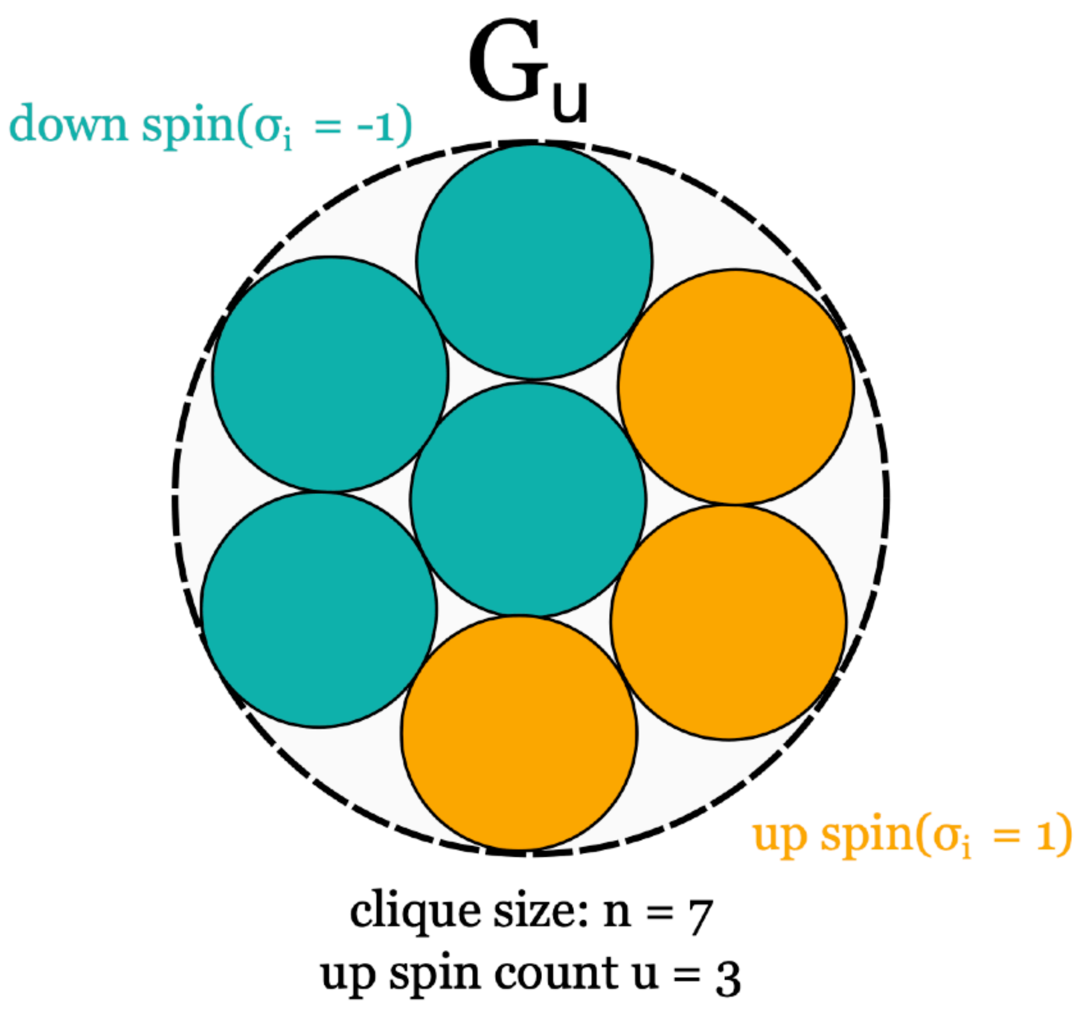


Deriving the Master Equation

The First Term

- Approximate master equations(AMES) are high accuracy approximations of binary state dynamics on networks[a]
- Occupation number :
  G_u the fraction of the system in a clique with u up spins.
- Example: up spin out flux : the rate at which down spins flip to up spins

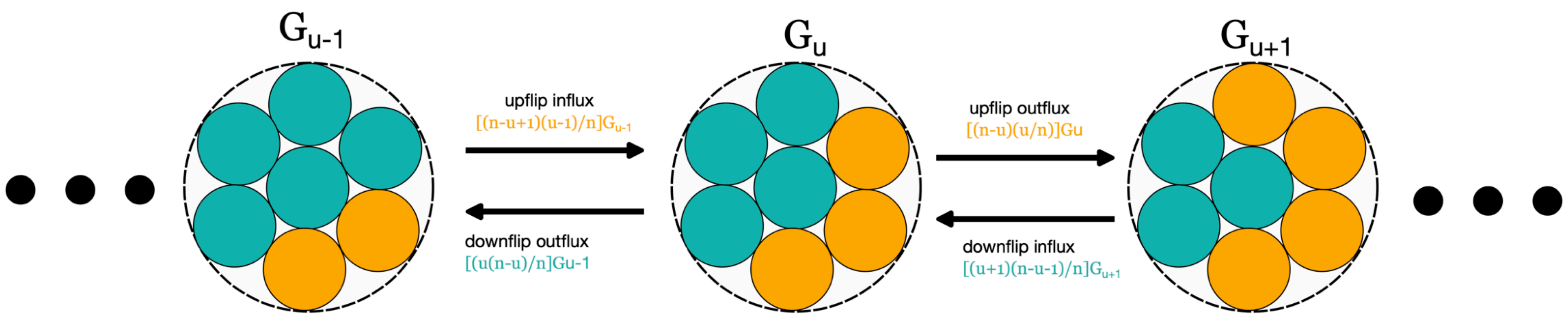
$$\underbrace{P(G_u \to G_{u+1})}_{\text{up spin out flux}} = \overbrace{G_u}^{\text{occupation number}} \underbrace{(n-u)}_{\text{of down spins in clique}} \left(\overbrace{\frac{u}{n}}^{\text{fraction of up spins in clique}}\right)^q$$



[a] Gleeson, 2011; Hébert-Dufresne et al., 2010; St-Onge et al., 2021.

Deriving the Master Equation

The Whole Shebang



Definition 3.1

Voter Model Master Equation for Constant Clique Size with Uncoupled Cliques

up spin in flux · *down spin in flux*

$$\frac{dG_u}{dt} = G_{u-1}\left[(n-u+1)\left(\frac{u-1}{n}\right)^q\right] + G_{u+1}\left[(u+1)\left(\frac{n-u-1}{n}\right)^q\right] - G_u\left[(n-u)\left(\frac{u}{n}\right)^q\right] - G_u\left[(u)\left(\frac{n-u}{n}\right)^q\right]$$

(1)

up spin out flux · *down spin out flux*

Definition 4.1

Moment Closure
The moment closure approximates the coupling between a group and surrounding groups

$$\rho_u(t) = \langle k_{ex} \rangle \frac{\sum_u G_u((n-u)\left(\frac{u}{n}\right)^q)}{\sum_u G_u(n-u)} \quad (2)$$

$$\rho_d(t) = \langle k_{ex} \rangle \frac{\sum_u G_u(u\left(\frac{n-u}{n}\right)^q)}{\sum_u G_u(u)} \quad (3)$$

Definition 4.2

Voter Model Master Equation for Constant Clique Size and Moment Closure

$$\frac{dG_u}{dt} = G_{u-1}\left[(n-u+1)\left(\frac{u-1}{n}\right)^q + \rho_u\right] + G_{u+1}\left[(u+1)\left(\frac{n-u-1}{n}\right)^q + \rho_d\right] - G_u\left[(n-u)\left(\frac{u}{n}\right)^q + \rho_u\right] - G_u\left[(u)\left(\frac{n-u}{n}\right)^q + \rho_d\right] \quad (4)$$

Solving For The Steady State

Definition 5.1

Detailed Balance In equilibrium, each elementary process is in equilibrium with its reverse process.

$$P(G_u \to G_{u+1}) = P(G_{u+1} \to G_u)$$

$$P(G_u \to G_{u-1}) = P(G_{u-1} \to G_u)$$

We know the recursion formula is

$$G_u = \frac{(n-u+1)\left[\rho + \left(\frac{u-1}{n}\right)^q\right]}{u[\rho + \frac{n-u}{n}]} G_{u-1}$$

So the formula for G_u is

$$G_u = \frac{1}{Z} \prod_{i=0}^{u} \frac{(n-u+1)\left[\rho + \left(\frac{u-1}{n}\right)\right]}{u[\rho + \frac{n-u}{n}]}$$

Where

$$Z = \sum_{u=1}^{N} \prod_{j=0}^{u} \frac{(n-j+1)\left[\rho + \left(\frac{j-1}{n}\right)\right]}{j[\rho + \left(\frac{j-1}{n}\right)]}$$

Linear Results

Linear Voter Model(q = 1)

Coexistence emerges as coupling increases

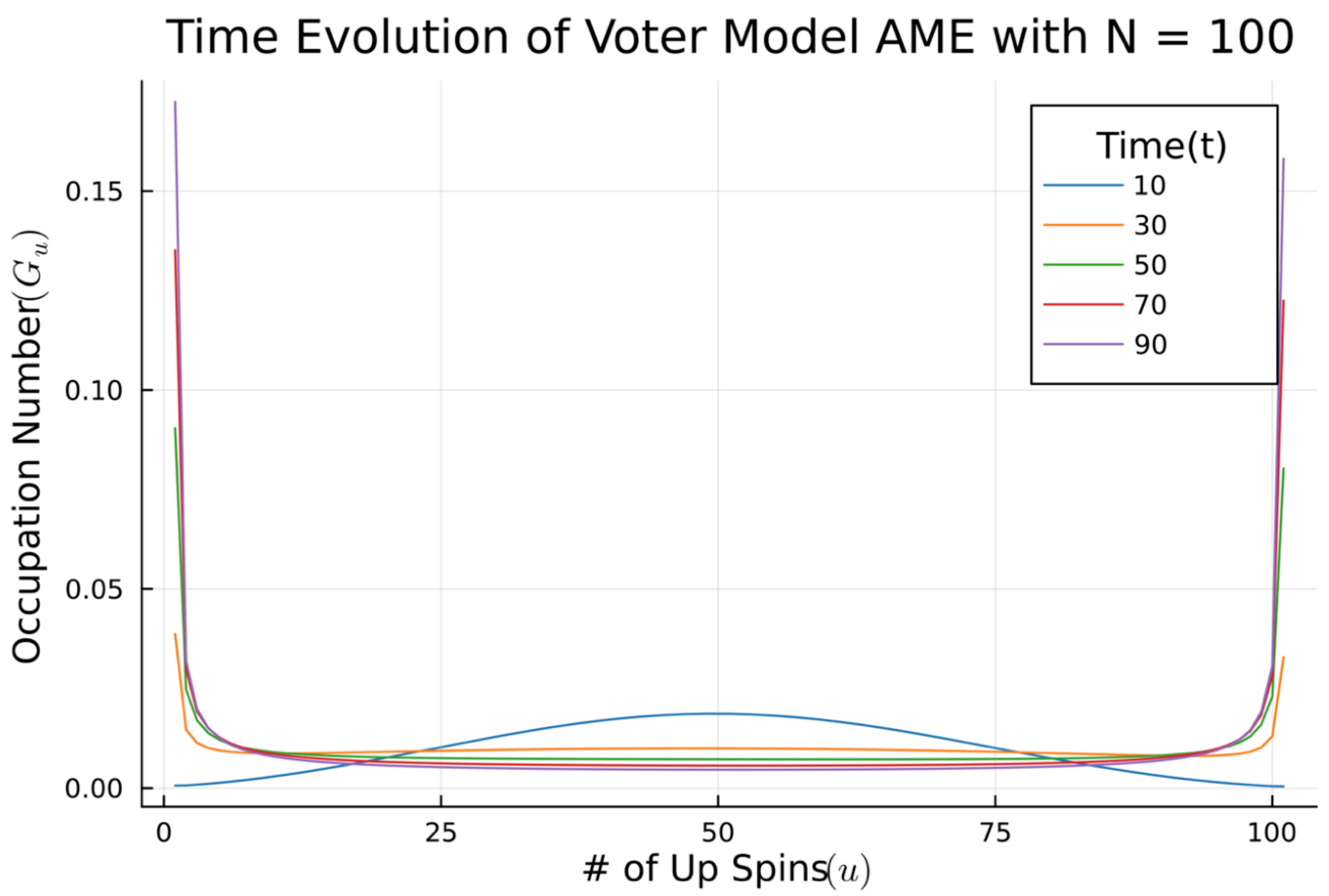


Figure. Time series of numerical integration of AME with $\rho = 0.0$ the distribution collapses two the two absorbing states

Coexistence emerges as coupling increases

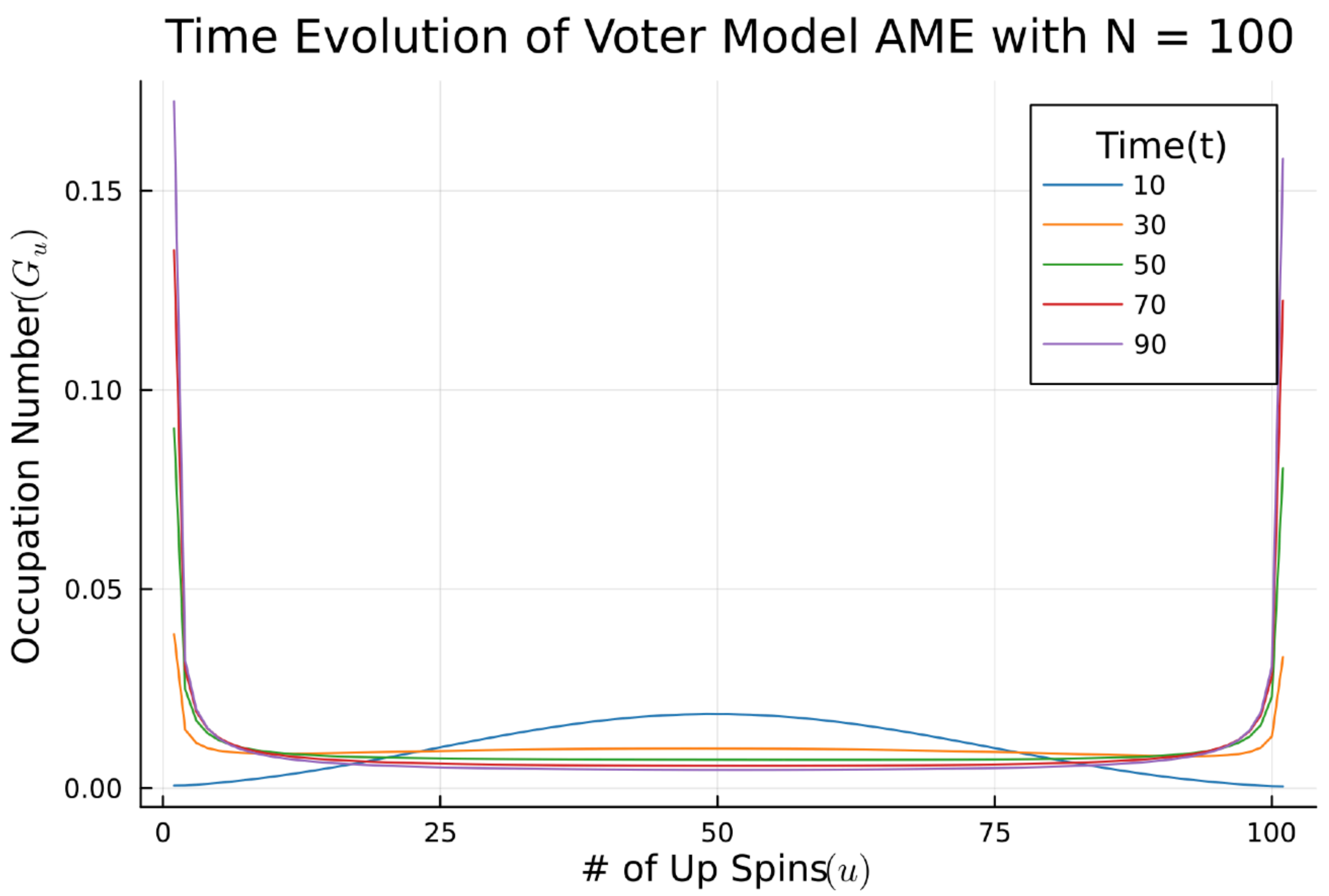


Figure. Time series of numerical integration of AME with $\rho = 0.0$ the distribution collapses two the two absorbing states

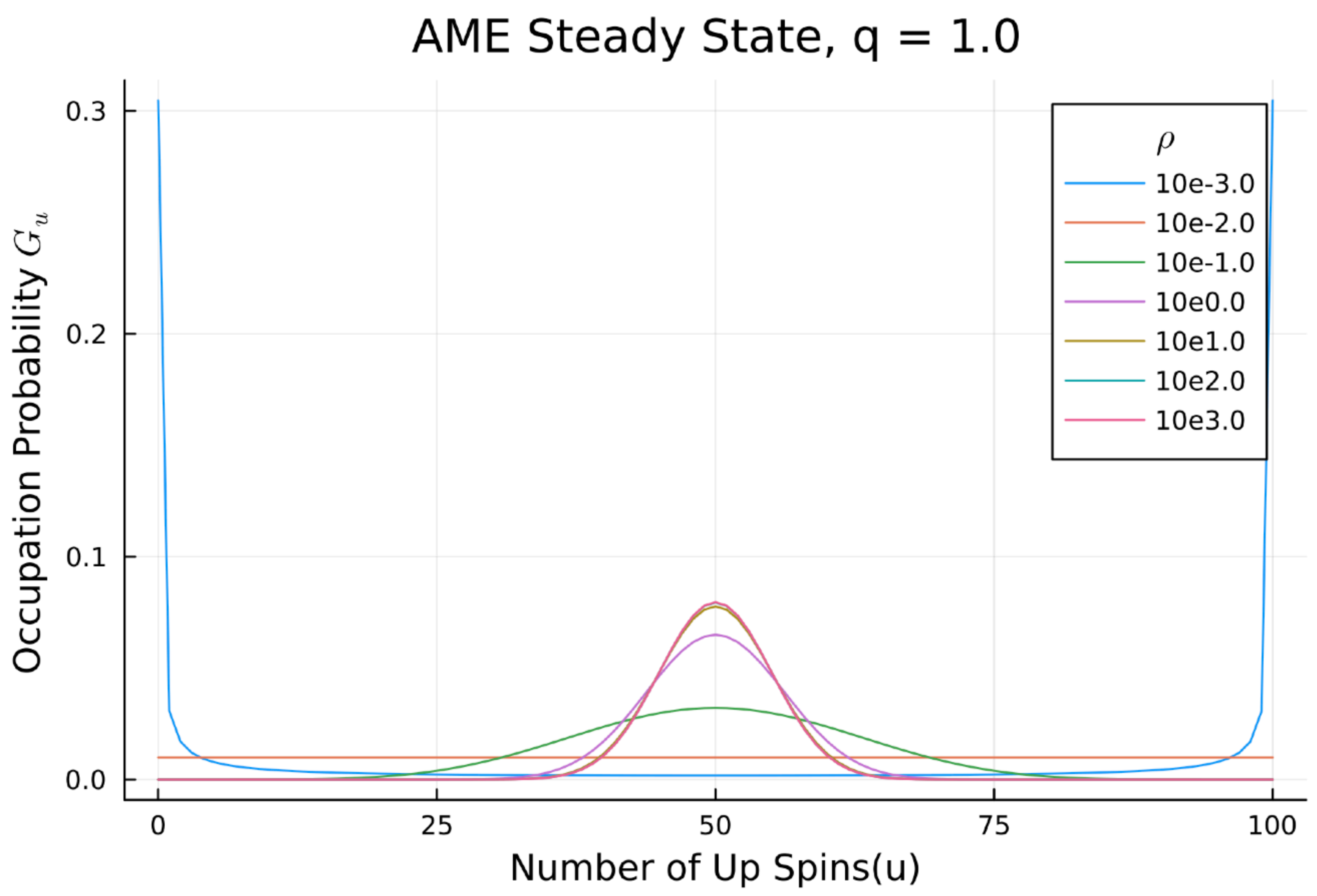


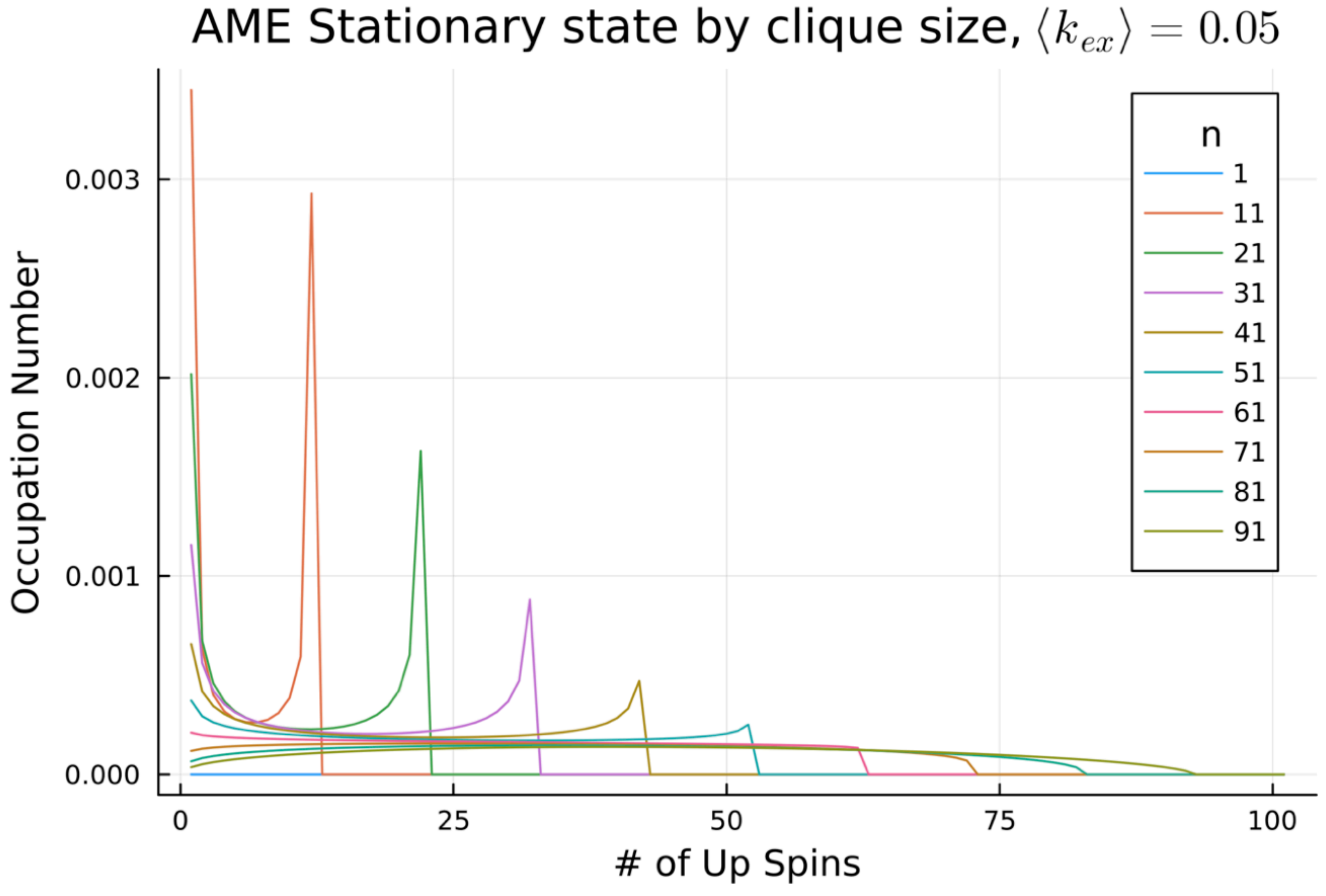
Figure. Steady state distribution for AMES as a function of ρ. Coexistence emerges as coupling increases

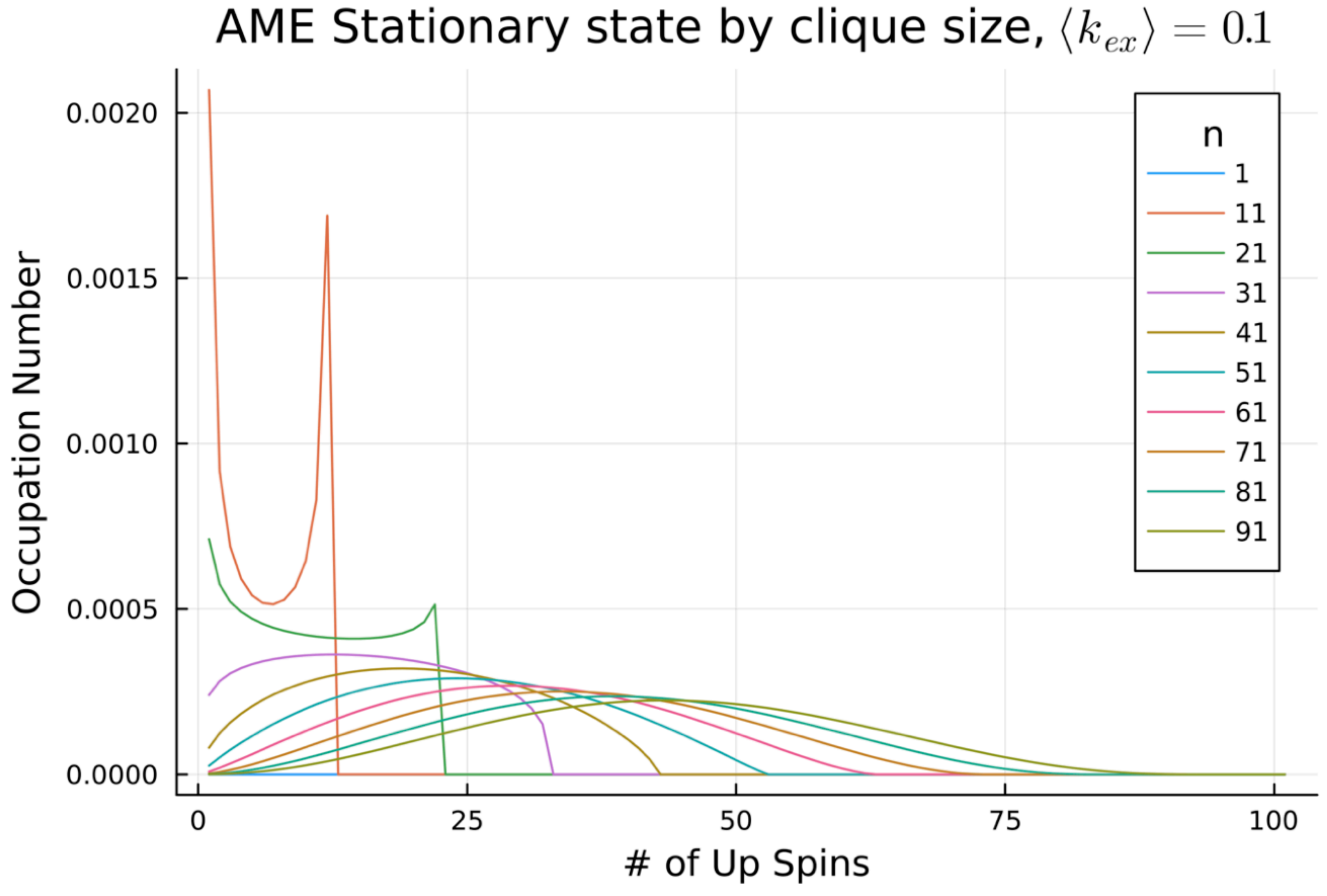
Linear Results

Heterogeneous Group Sizes

Larger cliques support coexistence at lower coupling strengths

At $\langle k_{ex} \rangle = 0.05$, coexistence only occurs at $n > 50$. At $\langle k_{ex} \rangle = 0.1$, coexistence occurs above $n > 20$

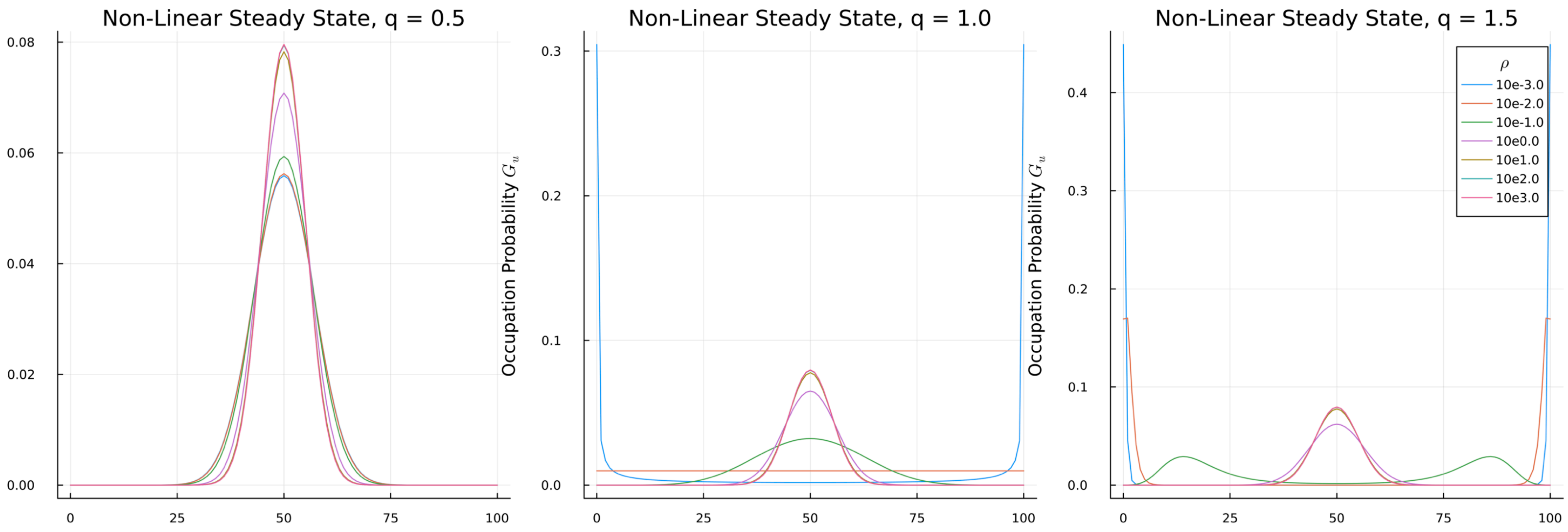




Non Linear Results

Conformist nodes($q > 1$) create stable minorities

- For $q < 1$ hipster nodes drive model towards coexistence
- For $q > 1$ conformist nodes create a stable minority

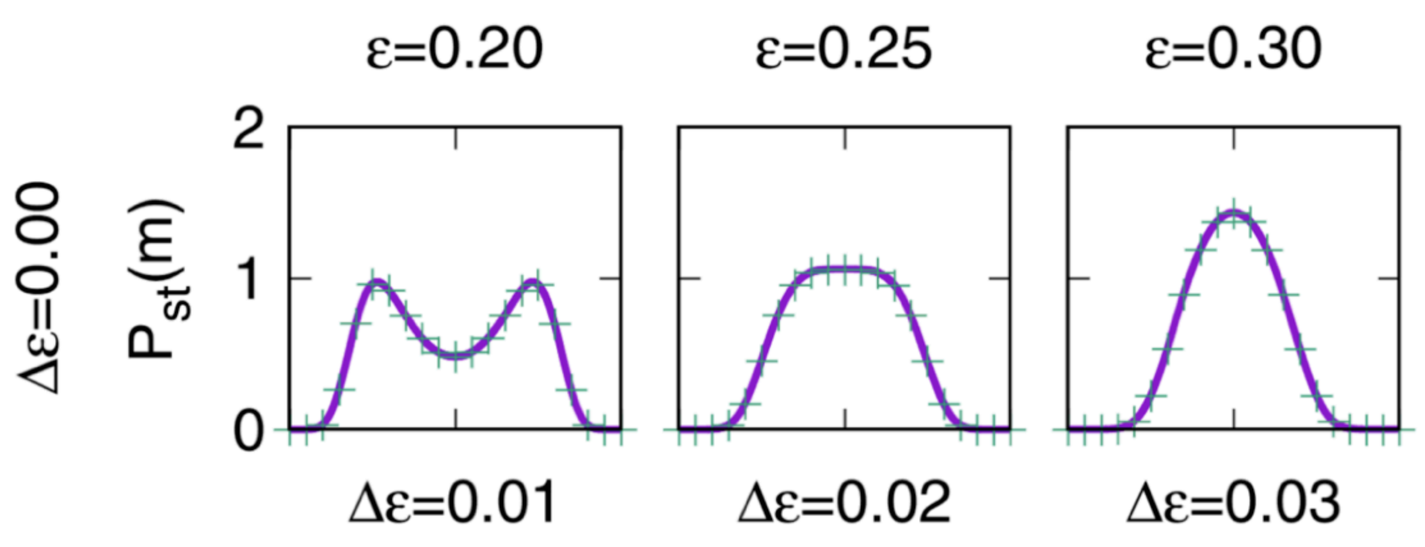


Noisy Non Linear Voter Model[a]

1. Give each node a random chance to flip, separate rates for each spin states a_0, a_1.
2. Total noise level $\epsilon \propto a_0 + a_1$
3. Noise asymmetry level $\Delta\epsilon \propto a_0 - a_1$

Steady State Distributions

1. Certain noise levels lead to emergence unimodal magnetization distribution
2. Others lead to bimodal distribution



[a] Peralta et al., 2018.

Island of Minority Coexistence

- Let's determine the possible states of the model by plotting the number of local maxima by parameter values
- For low ρ, the critical transition of the original q voter model remains the same.
- At higher couplings $\rho > 0.01$, consensus states become impossible. We see a bimodal distribution where stable minorities coexist within cliques.