Modeling Network Dynamics with Generating Functions For Fun and Profit

The Voter Model

What is a Graph?

A set of vertices V

A set of edges $E \subset \{\{i,j\} \,|\, i,j \in V \quad \text{and} \quad x \neq y\}$

A graph is defined as tuple of vertices and edges: $G = (V, E)$

Voter Model:

Assign a value $S = \{\sigma_i \in A, B \,|\, i \in V\}$

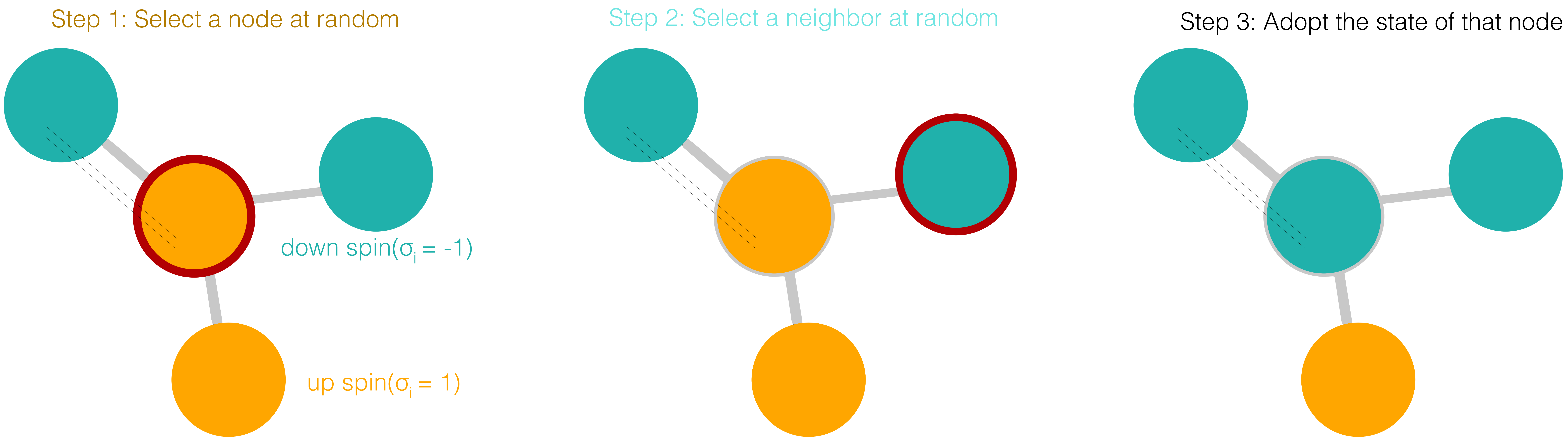
σ_i Represents one of two opinions on an issue:

- Democrat or Republican
- Coke of Pepsi

Voter Model Update Rule

Transition Rate $\propto$ portion of disagreeing neighbors

1. Choose a node $i \in V$ with state $\sigma_i = \{A, B\}$ uniformly at random
2. Chose a node from the neighbor $j \in \partial_i$ where $\partial_i = \{j \in V \,|\, (i,j) \in E\}$
3. Node i adopts the state of node j: $\sigma_i = \sigma_j$
4. Repeat until convergence.



Hypergraphs

Hyper-edges consist of arbitrary subsets of vertices $E = \{e \subseteq V\}$

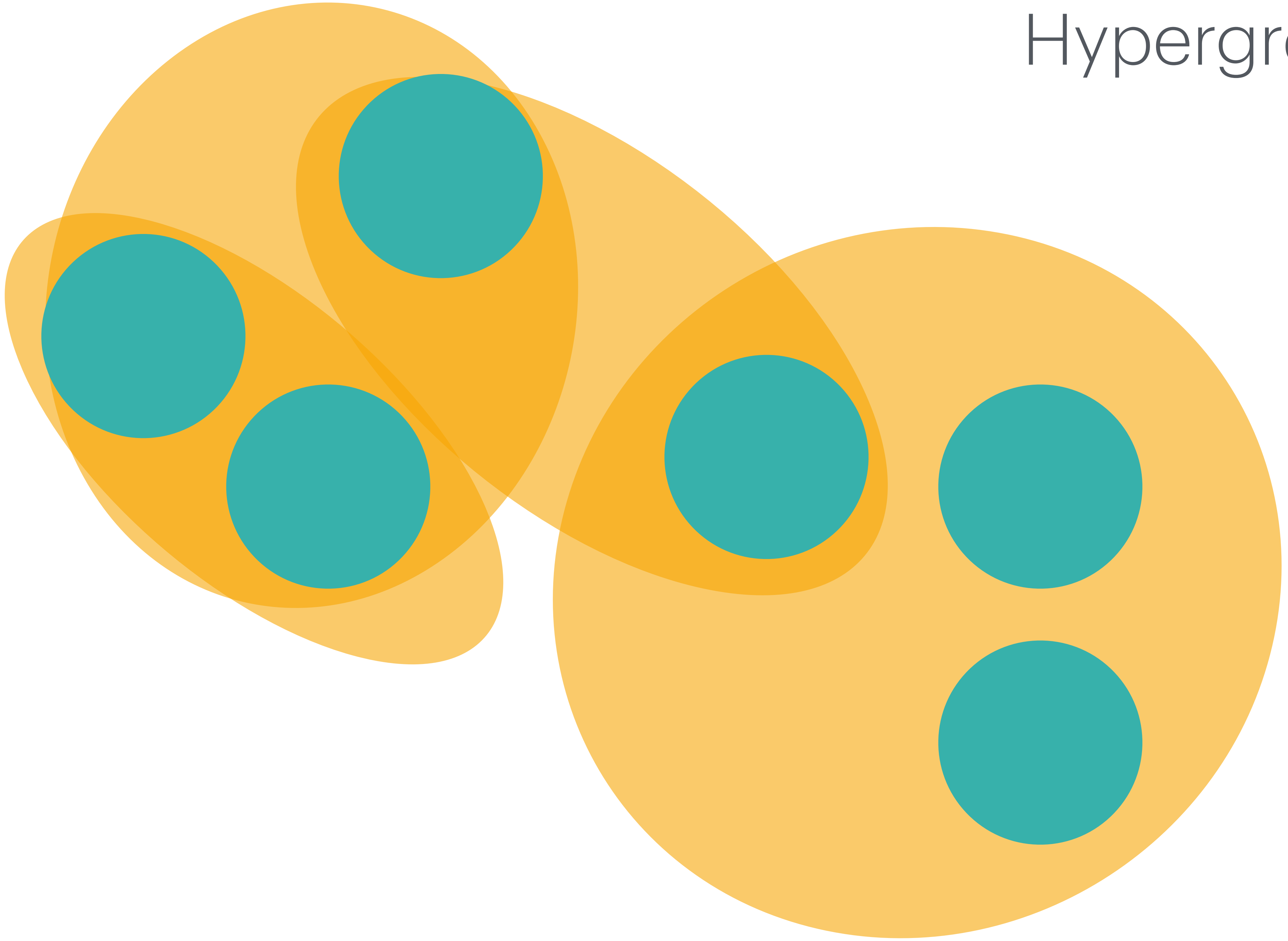
A **hypergraph** is a tuple of hyper-edges and vertices $H = (V, E)$

A **hyper-graph** can also be thought of as a **bipartite graph**

Why Hyper-graphs?

Hyper-graphs encode **group structure!**
Important to understand social dynamics

Hypergraph



Bipartite Graph

Vertices represent **individuals** with a given opinion: A,B

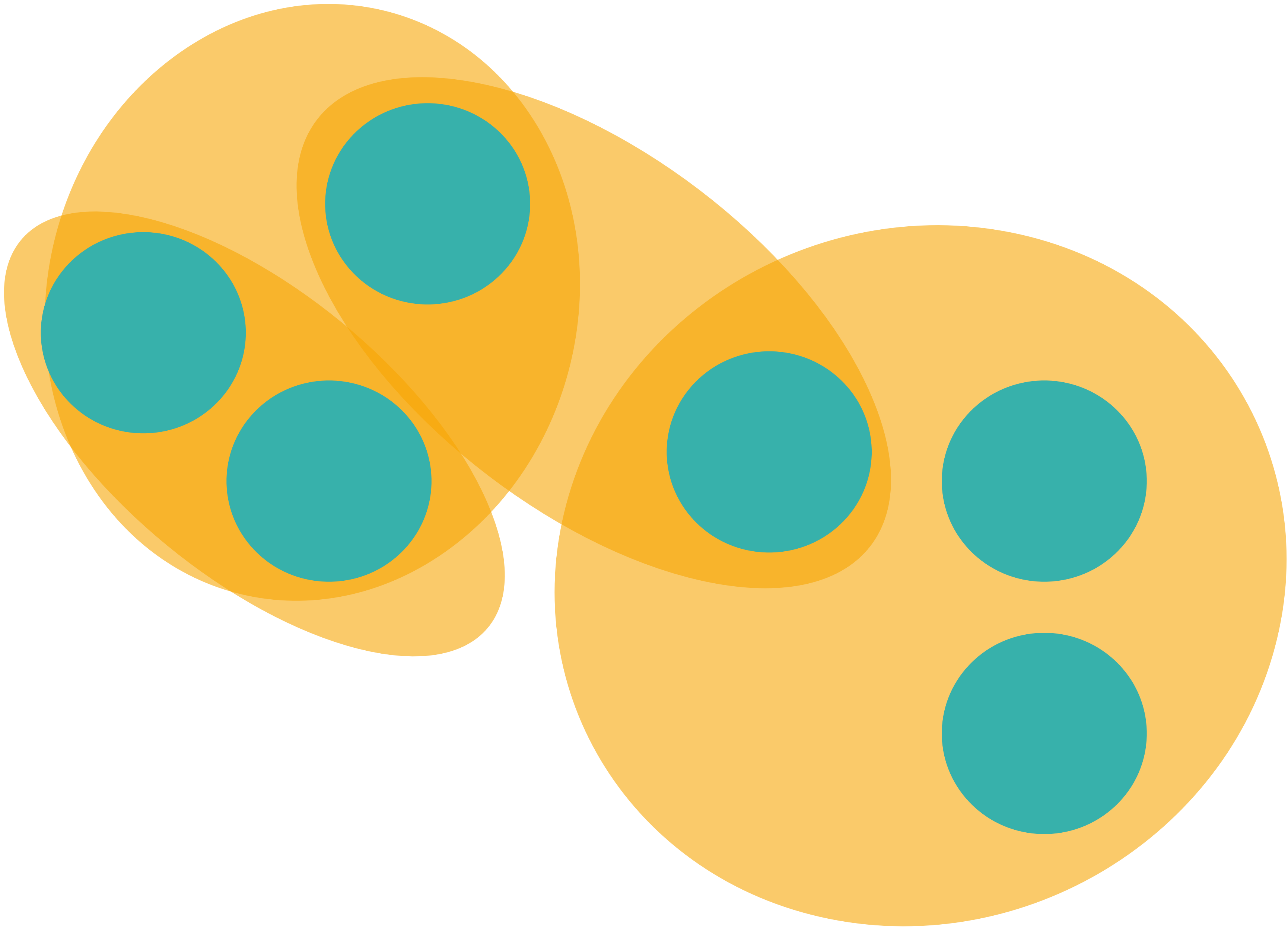
Hyper-edges represent **cliques:** social groups.

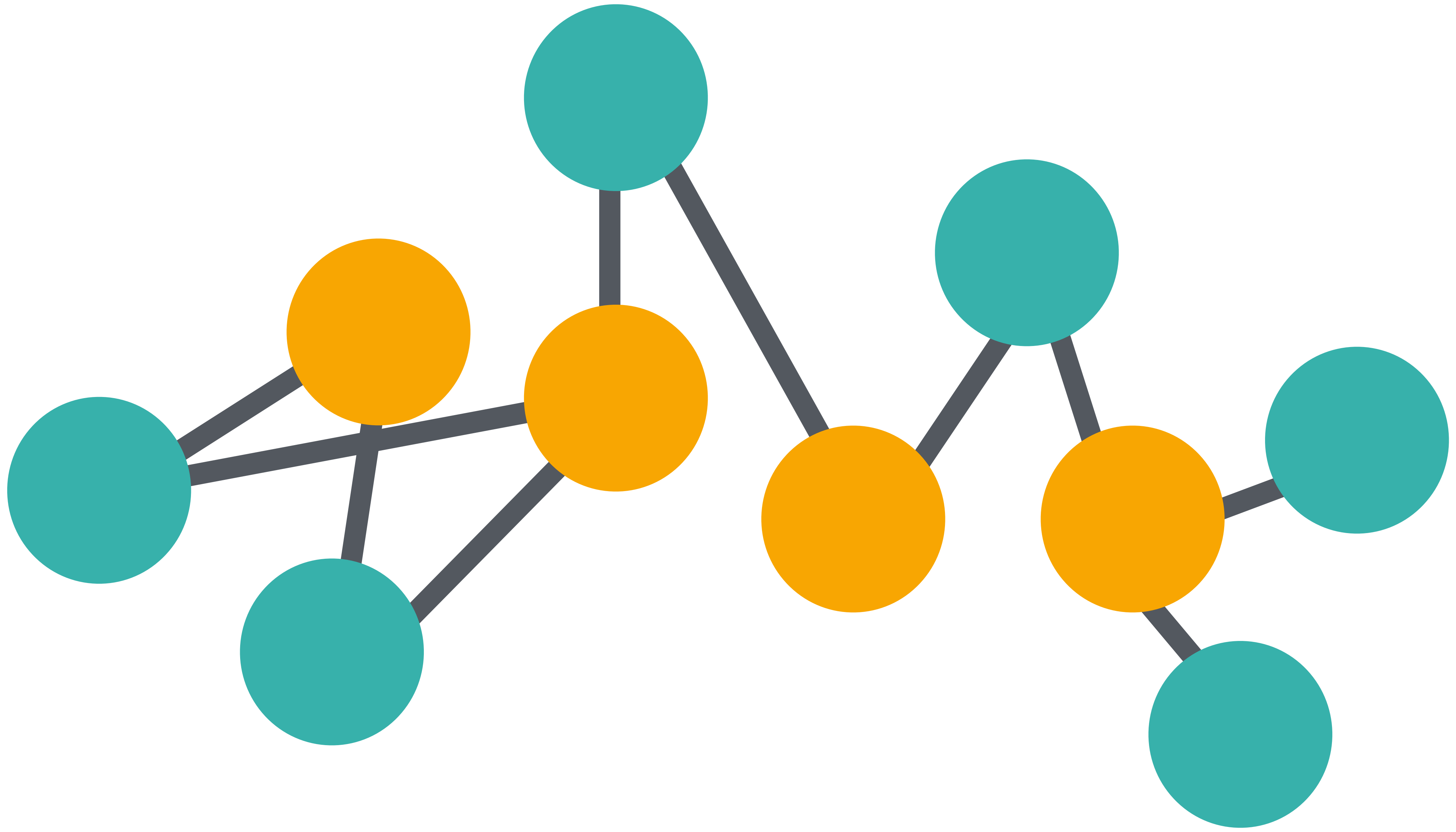
We define deg$(i) = |\partial_i|$

We characterize a our model with two degree distributions

g_m: **clique size distribution**: The degree distribution of individuals per clique

f_n = **clique membership distribution:** The distribution of cliques per node





For a degree distribution p_n with PGF $G(x) = \sum_{n=1}^{\infty} p_n x^n$

The mean is the derivative of the PGF evaluated at 1:

$\langle k \rangle = G_0'(1)$

The nth moment is $\langle k^n \rangle = \left(x \frac{d}{dx}\right)^n G_0(x)\,|_{x=1}$

Pick a node - follow a random edge. The **edge sampling degree distribution** is generated by $G_1(x) \; \frac{G_0(x)}{G_0'(1)}$

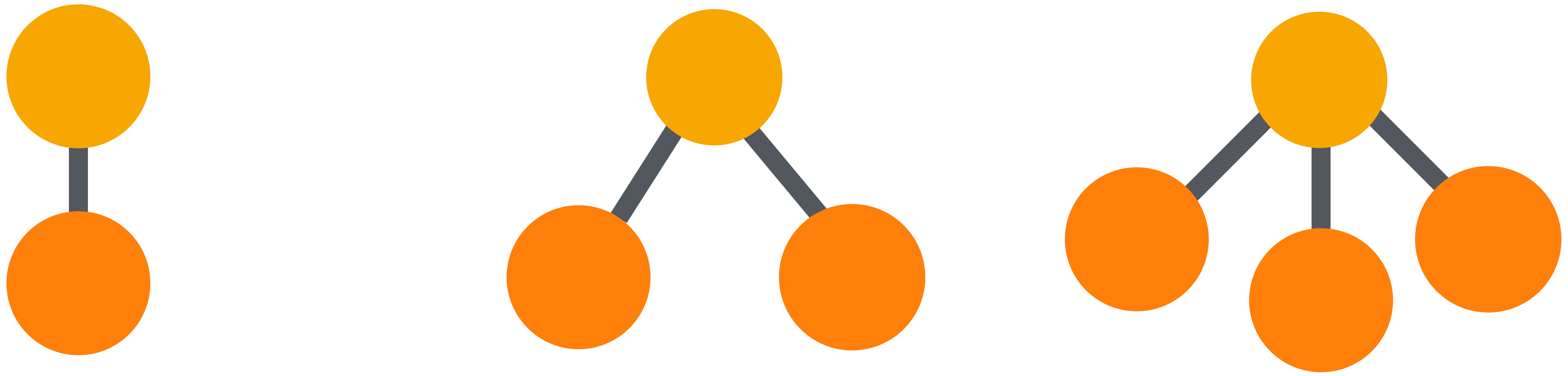
Distribution of **second neighbors**: $G_0(G_1(x))$

Multivariate PGF

$$G(x, y) = \sum_{n,n}^{\infty} P_{m,n} x^m y^n$$

$G_0(x)$: Select a **node** uniformly at random : p_n: the probability it has n neighbors

$$G_0(x) = p_1 x^1 + p_2 x^2 + p_3 x^3 \ldots p_k x^k$$



Cool Properties

The **mean degree** is the **derivative of the PGF evaluated at 1**: $\langle k \rangle = G_0'(1)$

The **nth moment** is the **nth derivative evaluated at 1** $\langle k^n \rangle = \left(x\frac{d}{dx}\right)^n G_0(x)\,|_{x=1}$

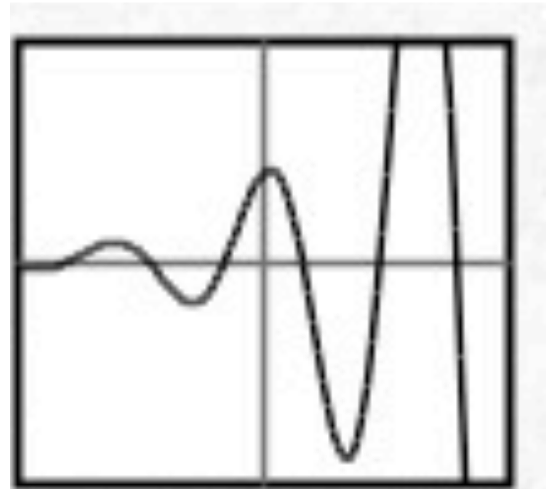
$g(x) = \sum g_m x^n$: for clique membership distribution

$F(x, y) = \sum A_{k,l} x^k y^l$: clique membership for A nodes

$G(x, y) = \sum B_{k,l} x^k y^l$: clique membership for B nodes

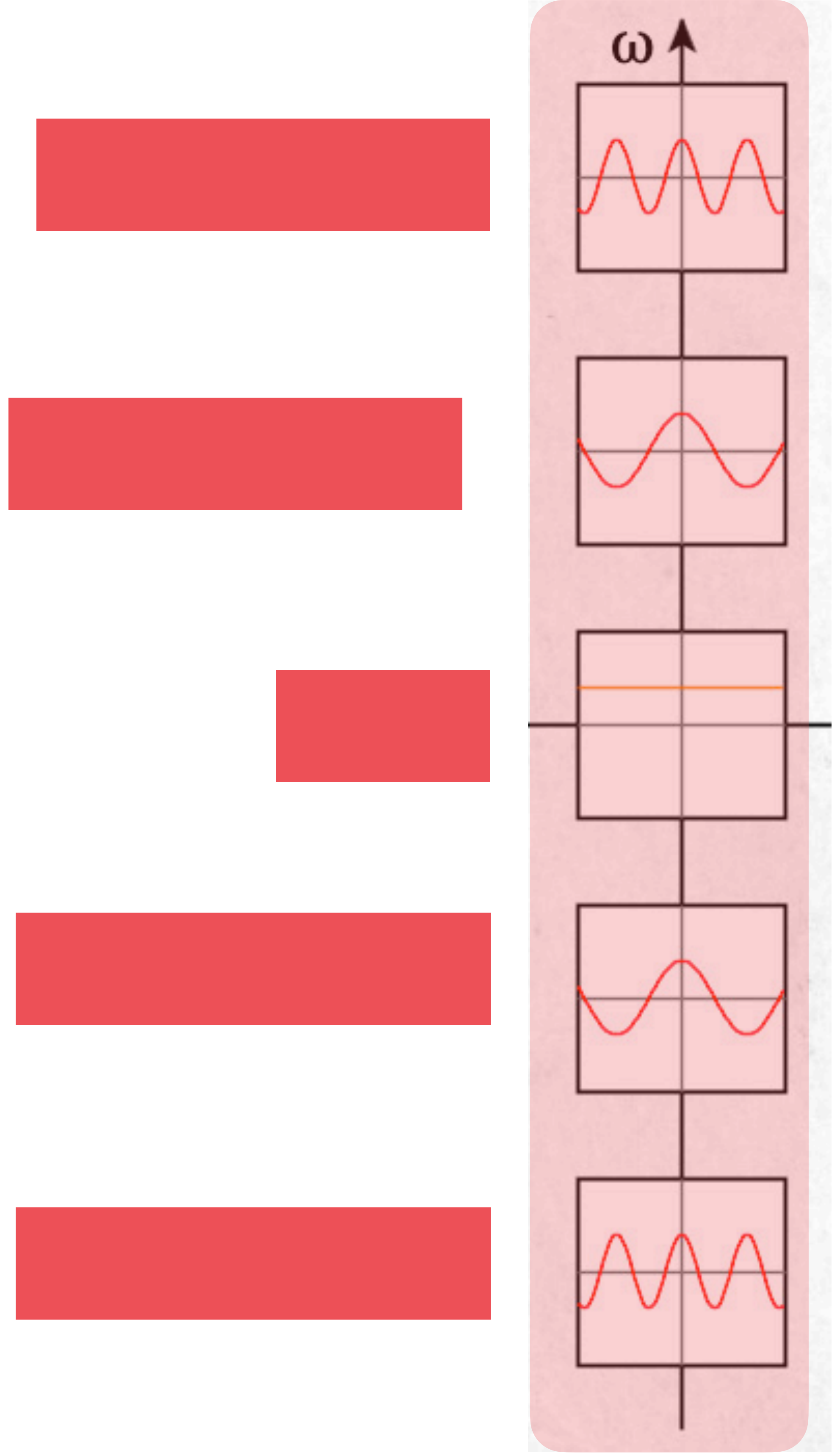
A **Fourier Transform** decomposes a function into a weighted sum of sinusoid functions

$f(t)$



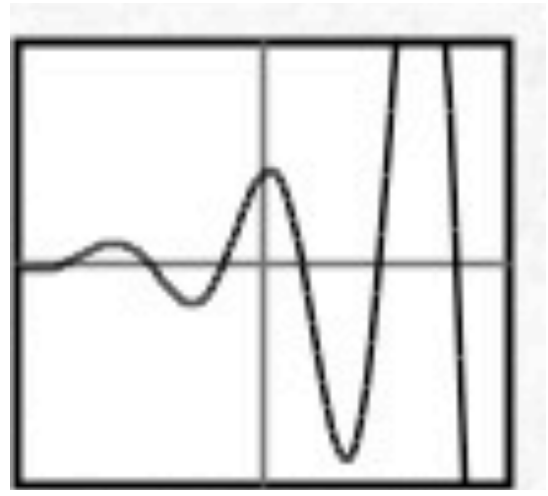
$$\mathscr{F}\{f\}(\omega) = \int f(x)e^{-i\pi\omega t}\, dx$$

$f(\omega)$

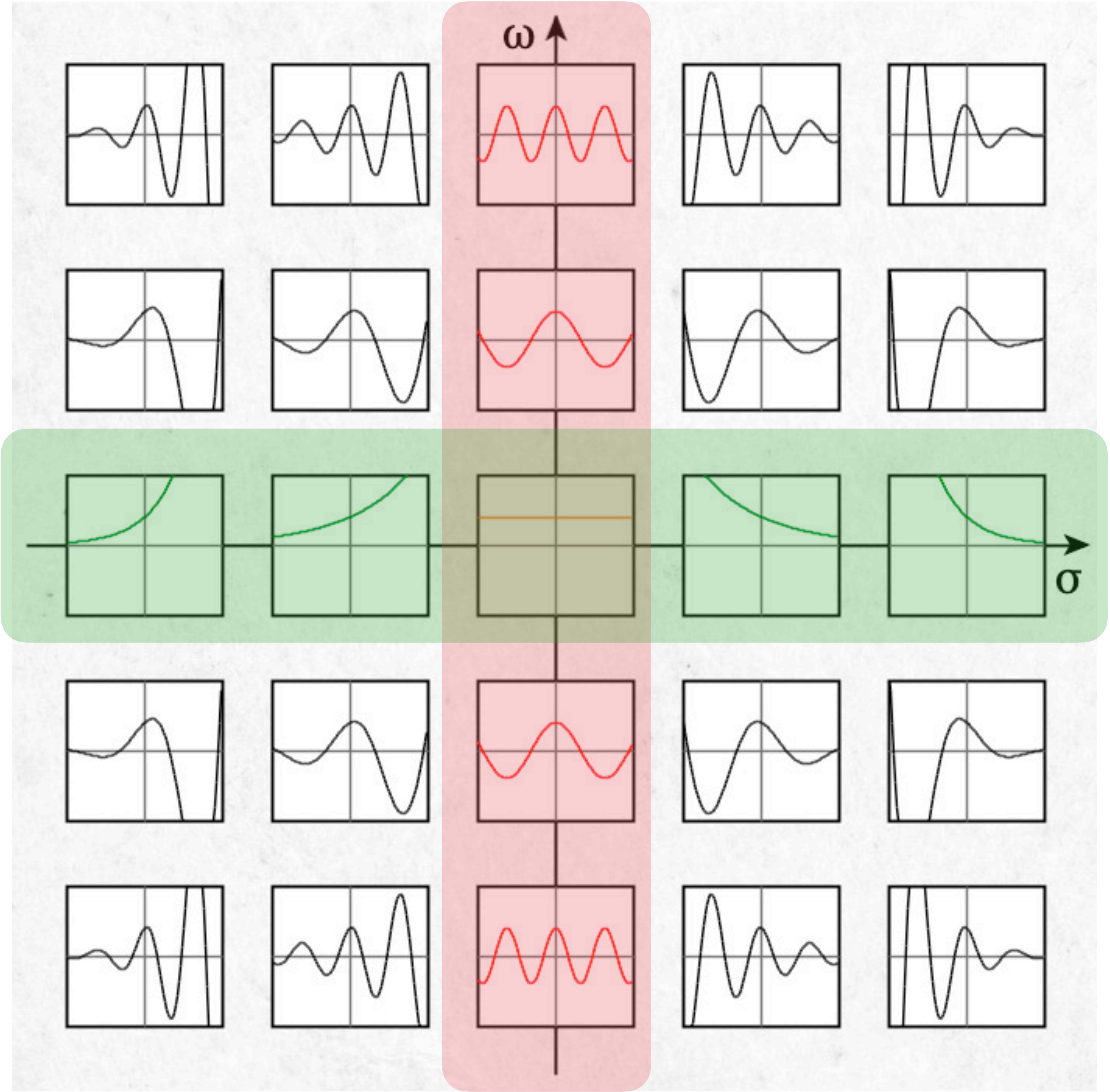


A **Laplace Transform** decomposes a function into sinusoids × decaying exponentials

$f(t)$



$$\mathscr{L}\{f\}(s) = \int f(t)e^{-st}\,dt$$



s is a complex number $s = \omega + i\sigma$

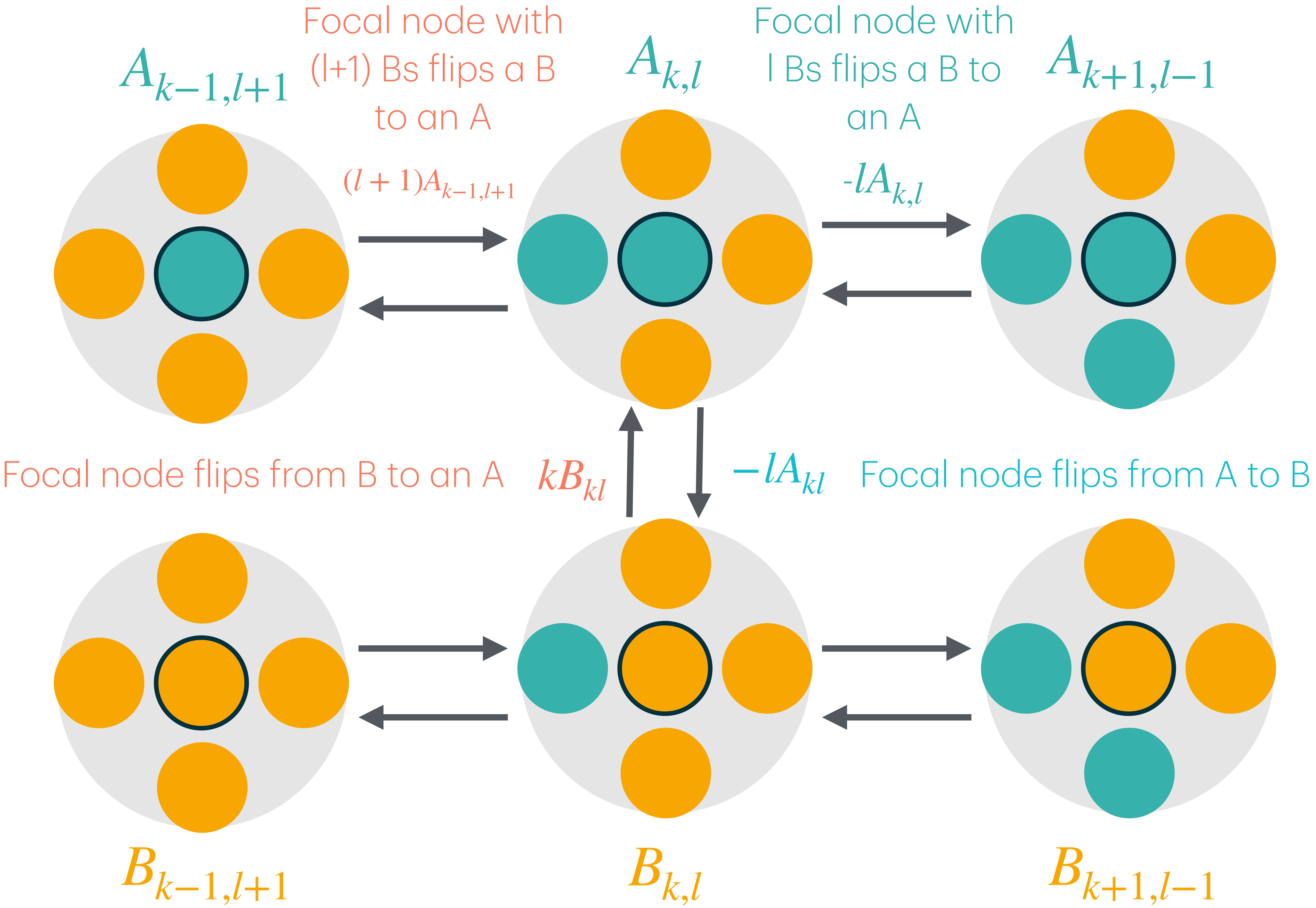
$$p_b = \frac{1}{n!}\frac{d^n}{dx^n}G(x)\,|_{x=0} = \frac{1}{2\pi i}\oint_\gamma \frac{G(x)}{z^{s+1}}dz$$

Coefficients of a power series be repeated differentiation

Can be expressed as a contour integral on the complex plane

$$p_b = \frac{1}{Nr^n}\sum_{m=0}^{N-1} G(re^{2\pi mi/N})e^{2\pi im/N}$$

For a polynomial this can be expressed as a discrete Fourier transform

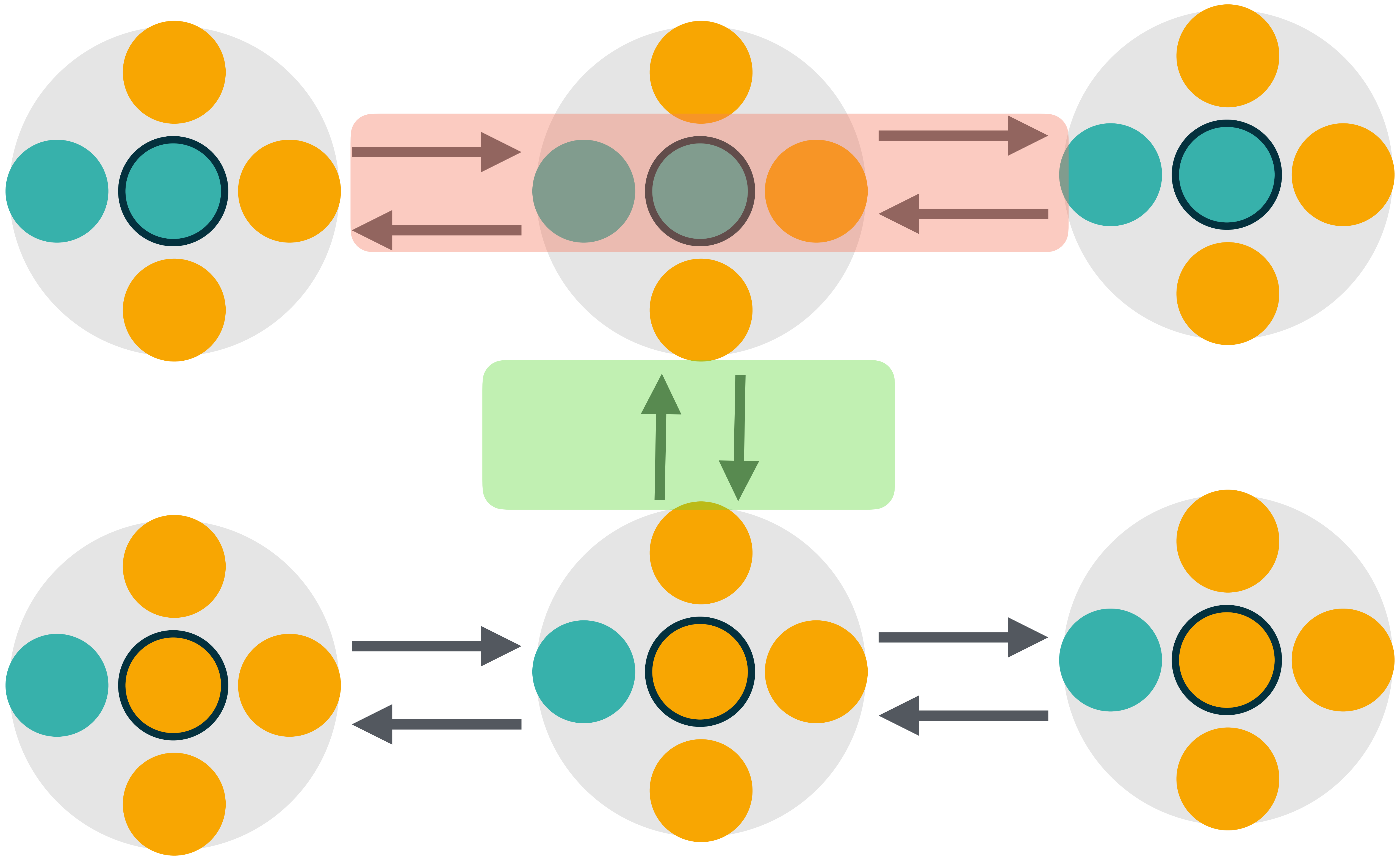

Focal node with
(l+1) Bs flips a B
to an A
$(l+1)A_{k-1,l+1}$
Focal node with
l Bs flips a B to
an A
$-lA_{k,l}$
$A_{k-1,l+1}$
$A_{k,l}$
$A_{k+1,l-1}$
Focal node flips from B to an A
kB_{kl}
$-lA_{kl}$
Focal node flips from A to B
$B_{k-1,l+1}$
$B_{k,l}$
$B_{k+1,l-1}$

$$2\frac{dA_{k,l}}{dt} = kB_{k,l} - lA_{kl}$$

Focal nodes gets flipped

$$(l+1)A_{k-1,l+1} - A_{k,l}$$

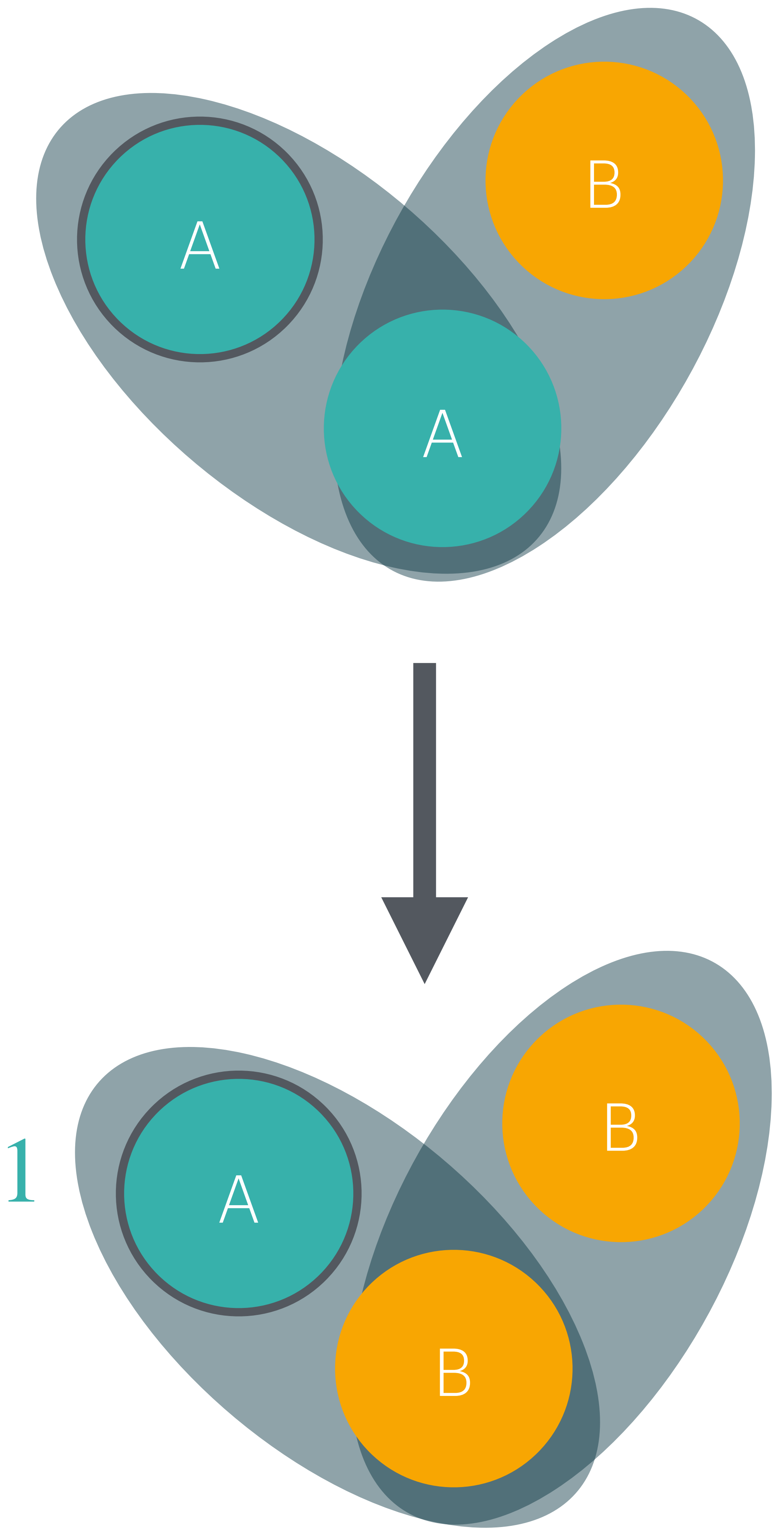
Focal node flips another node



Interclique Interactions - Moment Closures

$A_{k,l}$

$A_{k-1,l+1}$



Avg # of A nodes with an A and B as neighbors

Avg. # of cliques-1

$$\propto g_1'(1)\frac{F_{xy}(x,1)}{F_x(1,1)} = \langle k_{ex} \rangle \frac{\sum_{k,l} lkA_{k,l}}{\sum_{k,l} kA_{k,l}}$$

Avg # of A nodes with an A and B as neighbors

Focal node(A) has an A neighbor flipped to a B by an **interaction outside focal clique**

$\propto$ **A** - **A** - **B** triplets/**A** - **B** edges

$$2\frac{dA_{k,l}}{dt} = kB_{k,l} - lA_{kl}$$

Focal nodes gets flipped

$$+((l+1)A_{k-1,l+1} - A_{k,l})$$

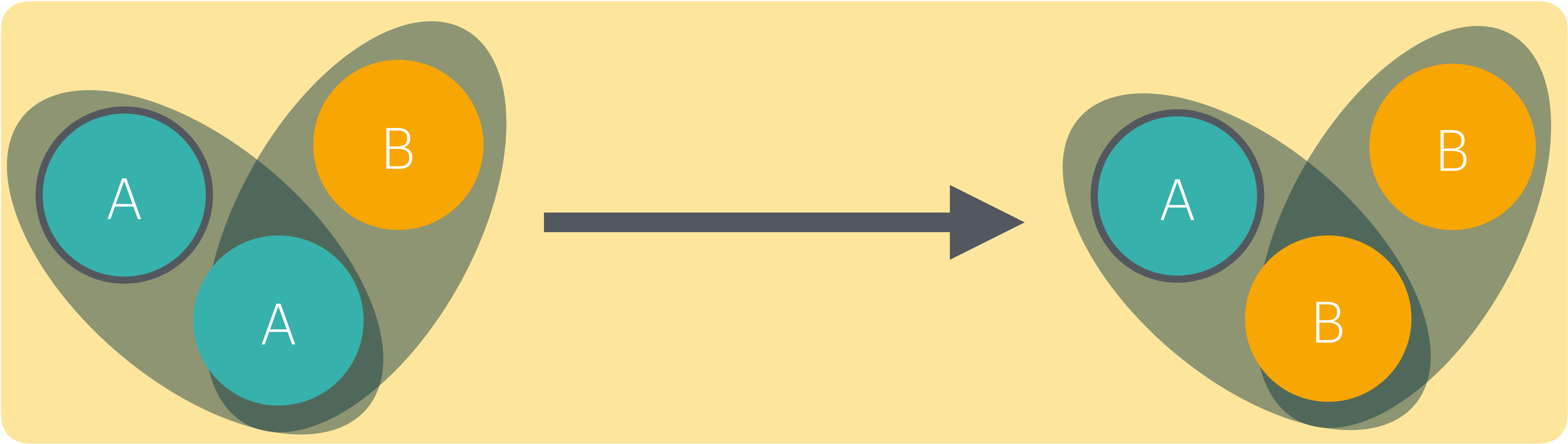
Focal node flips another node

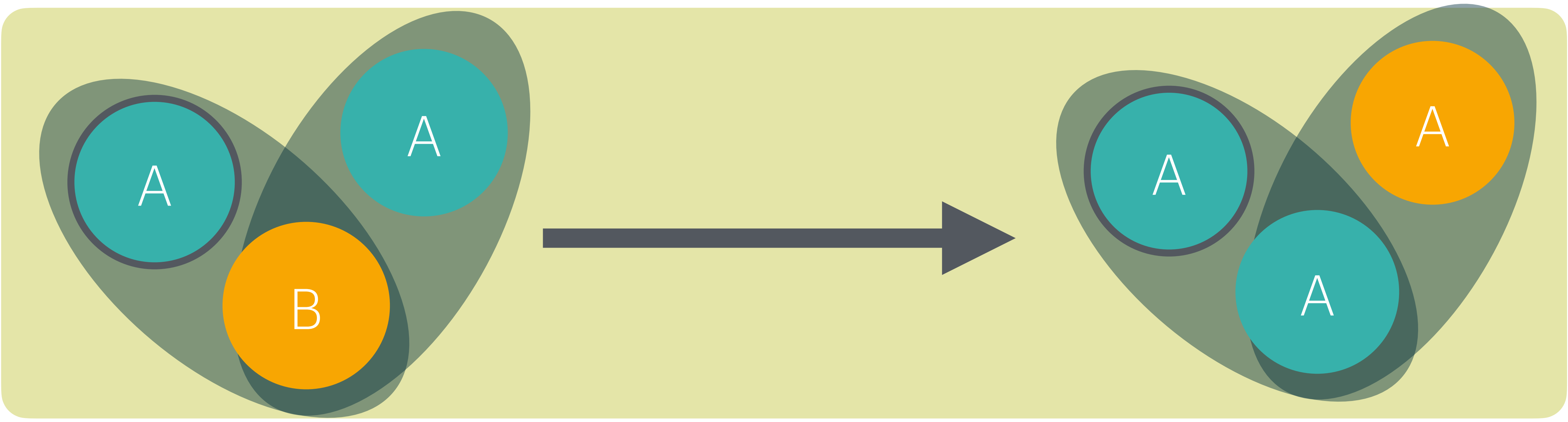
$$+g_1'(1)\frac{F_{xy}(x,1)}{F_x(1,1)}\big((l+1)A_{k-1,l+1} - lA_{k,l}\big)$$

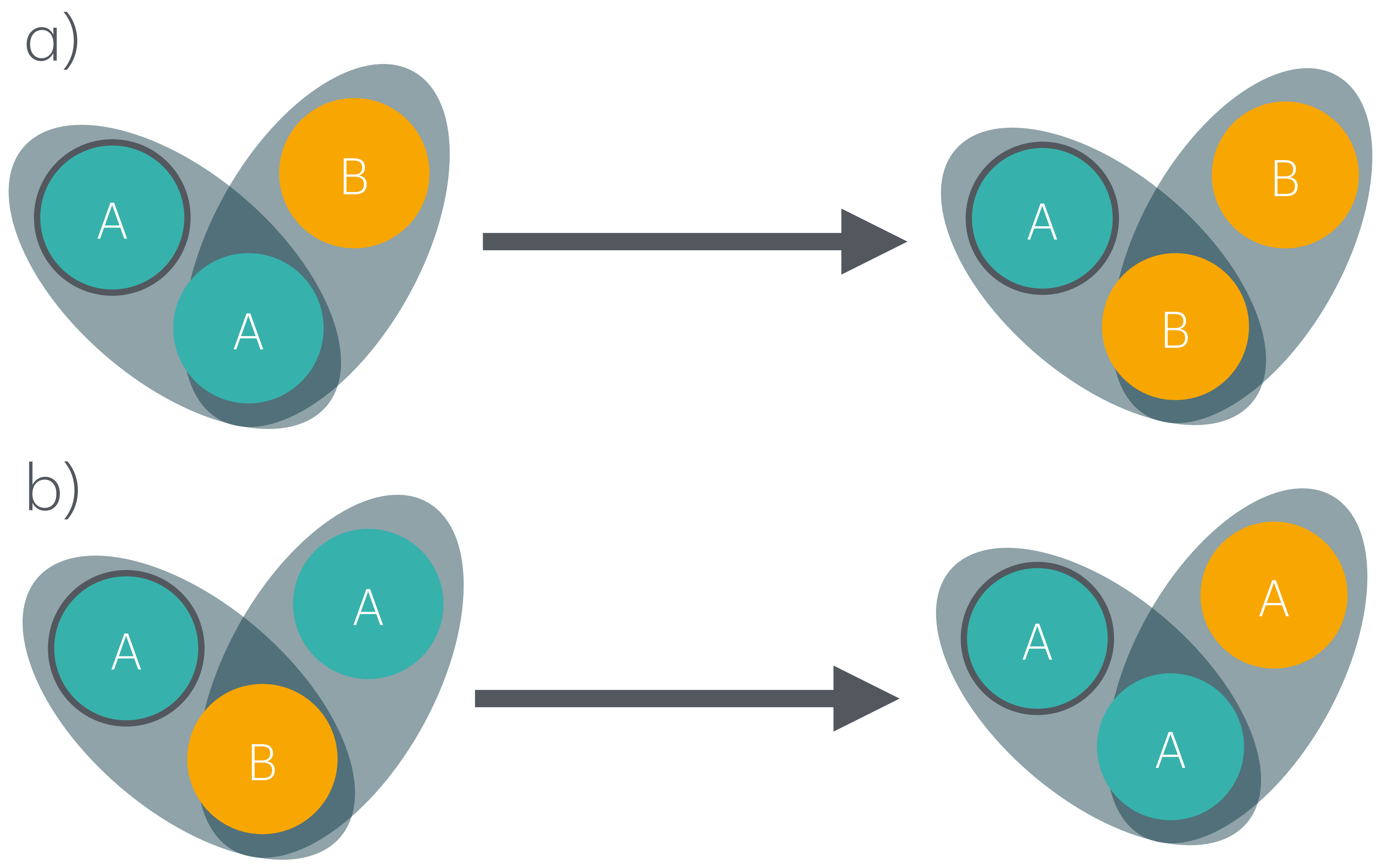
Focal node has neighbor flipped to a B by an interaction outside focal clique

$$+g_1'(1)\frac{G_{yx}(x,1)}{G_y(1,1)}\big((l+1)A_{k+1,l-1} - lA_{k,l}\big)$$

Focal node has neighbor flipped to a A by an interaction outside focal clique







a)
A
A
B
A
B
B
b)
A
A
B
A
A
A

$$\frac{dA_{k,l}}{dt} = \frac{1}{2}(kB_{k,l} - lA_{kl}) + \frac{1}{2}((l+1)A_{k-1,l+1} - lA_{k,l}) + \frac{1}{2}g_1'(1)\frac{F_{xy}(x,1)}{F_x(1,1)}\left((l+1)A_{k-1,l+1} - lA_{k,l}\right) + \frac{1}{2}g_1'(1)\frac{G_{yx}(x,1)}{G_y(1,1)}\left((k+1)A_{k+1,l-1} - kA_{k,l}\right)$$

$$\frac{dA_{k,l}}{dt} = \frac{1}{2}((l+1)A_{k-1,l+1} - A_{k,l}) + \frac{\phi}{2}((l+1)A_{k-1,l+1} - lA_{k,l}) + \frac{\psi}{2}\left((k+1)A_{k+1,l-1} - kA_{k,l}\right)$$

Approximate

$$lA_{k,l} \approx kB_{k,l}$$

$$\frac{F_{xx}(x,1)}{F_x(1,1)} \approx \frac{G_{yx}(1,y)}{G_y(1,1)}$$

Define

$$\phi = g_1'(1)\frac{F_{xy}(x,1)}{F_x(1,1)}$$

$$\psi = g_1'(1)\frac{F_{yx}(x,1)}{F_y(1,1)}$$

The time derivative of the generating function can be expressed as the PDE of the PGF

$$\frac{\partial F(x,y,t)}{\partial t} = \sum_{k,l}^{\infty} \frac{dA_{k,l}}{dt}$$

Start with an infinite system of ODEs

$$\frac{\partial F(x,y,t)}{\partial t} = \frac{1}{2}\sum_{kl}[(l+1)A_{k-1,l+1} - lA_{k,l}]x^k y^l + \frac{\phi}{2}\sum_{kl}[(l+1)A_{k-1,l+1} - lA_{k,l}]x^k y^l + \frac{\psi}{2}\sum_{kl}\left[(k+1)A_{k+1,l-1} - kA_{k,l}\right]x^k y^l$$

$$F_t = \frac{1}{2}\psi[y-x]F_x + \frac{1}{2}[(1+\phi)(x-y)]F$$

End with one quasilinear PDE

Rectify indices and express terms as derivatives of PGF

$$\sum_{kl}[(l+1)A_{k-1,l+1} - lA_{k,l}]x^k y^l = xF_y - yF_y$$

We can rewrite the equation in the form of a Liouville Equation

$$\partial_t F = L[x, y; \partial_x, \partial_y] F ,$$

With a **Liouville Operator**

$$L = \frac{1}{2} \psi (y - x) \partial_x + \frac{1}{2} (1 + \eta) (x - y) \partial_y .$$

We can replace our **auxiliary variable** and its derivatives with **second quantized operators**

$$x \to a_A^\dagger$$

Auxillary variable → **creation operator**

$$\partial_x \to a_A$$

Partial derivative to **annihilation operator**

General first order quasilinear PDE $a(t,x,y)F_t + b(x,y,t)F_x + c(x,y,t)F_y = d(x,y,t)$

Describe solution surface $F(x,y,t)$ parametrically in (t,x,y,F) space

PDE can be written as dot product $(a,b,c,d) \cdot (F_t, F_x, F_y, -1) = 0$

$(Q_t, Q_x, Q_y, -1)$ is **normal to solution surface** $F(x,y,t)$

So (a,b,c,d) is orthogonal to normal vector => **it is tangent to the solution surface**

So curves $(t(s), x(s), y(s), Q(s))$ will always be tangent to the surface IF they satisfy

$$(dt/ds, dx/ds, dy/ds, dQ/ds) = (a,b,c,d)$$

Our Quasi Linear PDE

$$F_t + \psi[y - x]F_x + [(1 + \phi)(x - y)]F_y$$

Characteristic Equations

$$\frac{dt}{ds} = a = 1$$

$$\frac{dx}{ds} = b = \frac{1}{2}\psi[y - x]$$

$$\frac{dy}{ds} = c = \frac{1}{2}(1 + \phi)[x - y]]$$

$$\frac{dF}{ds} = d = 0$$

$$\frac{dx}{ds} = \frac{1}{2}\psi[x - y]$$

$$\frac{dy}{ds} = \frac{1}{2}(1 + \phi)[y - x]$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{where} \quad A = \frac{1}{2}\begin{bmatrix} \psi & -\psi \\ -(1+\phi) & 1+\phi \end{bmatrix}$$

A has eigenvalues and eigenvectors

$$\lambda_1 = 0 \quad \nu_1 = (1,1) \quad \text{and} \quad \lambda_2 = (1 + \phi + \psi) \quad \nu_2 = (\frac{-\psi}{1+\phi}, 1)$$

$$\begin{pmatrix} x(s) \\ y(s) \end{pmatrix} = \alpha_1 \begin{pmatrix} x(s) \\ y(s) \end{pmatrix} = \alpha_2 e^{(1+\phi+\psi)s} \begin{pmatrix} \frac{-\psi}{1+\psi} \\ y(s) \end{pmatrix}$$

$$\begin{pmatrix} x(s) \\ y(s) \end{pmatrix} = e^{As} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

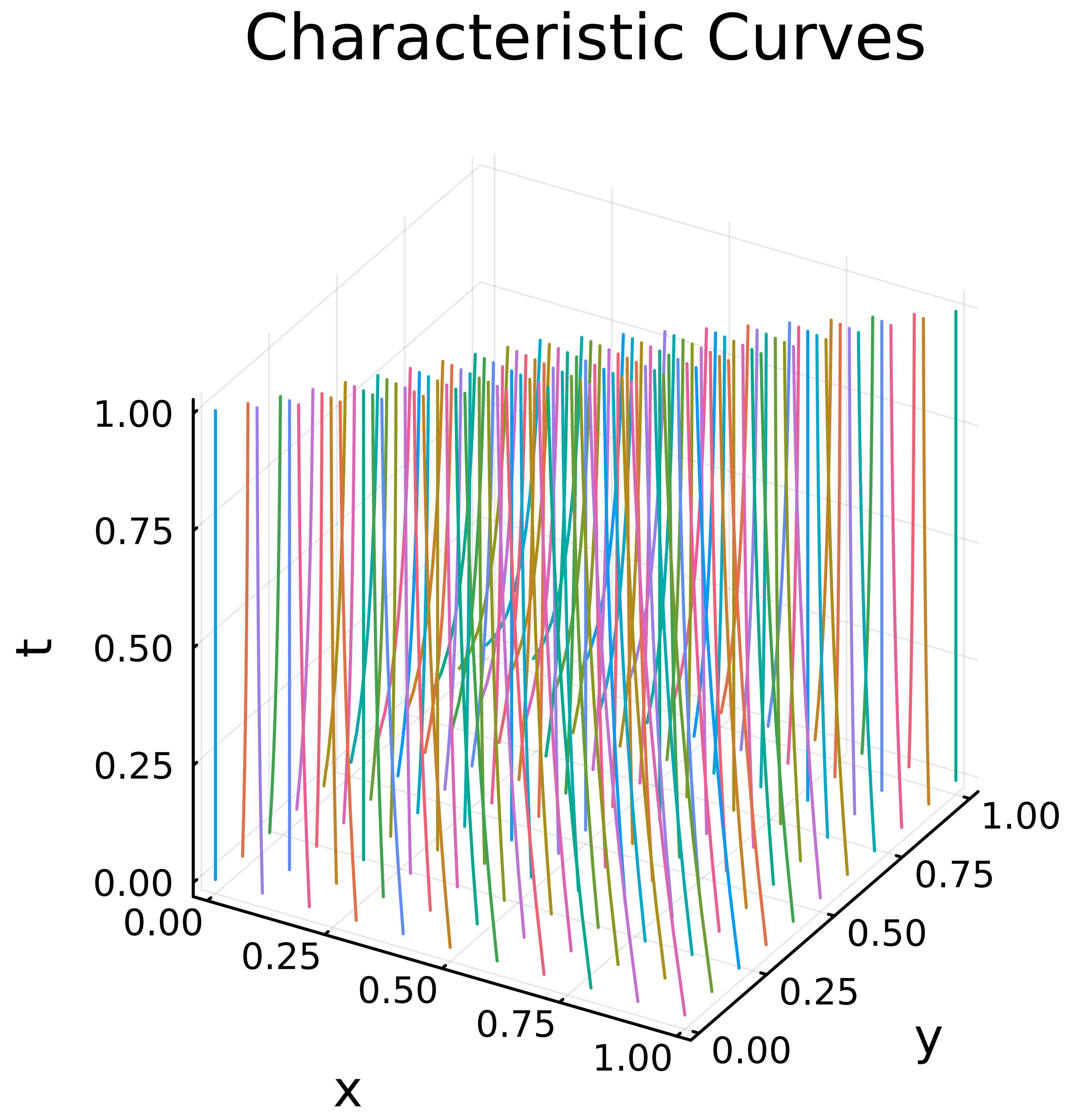
$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = e^{-As} \begin{pmatrix} x \\ y \end{pmatrix}$$

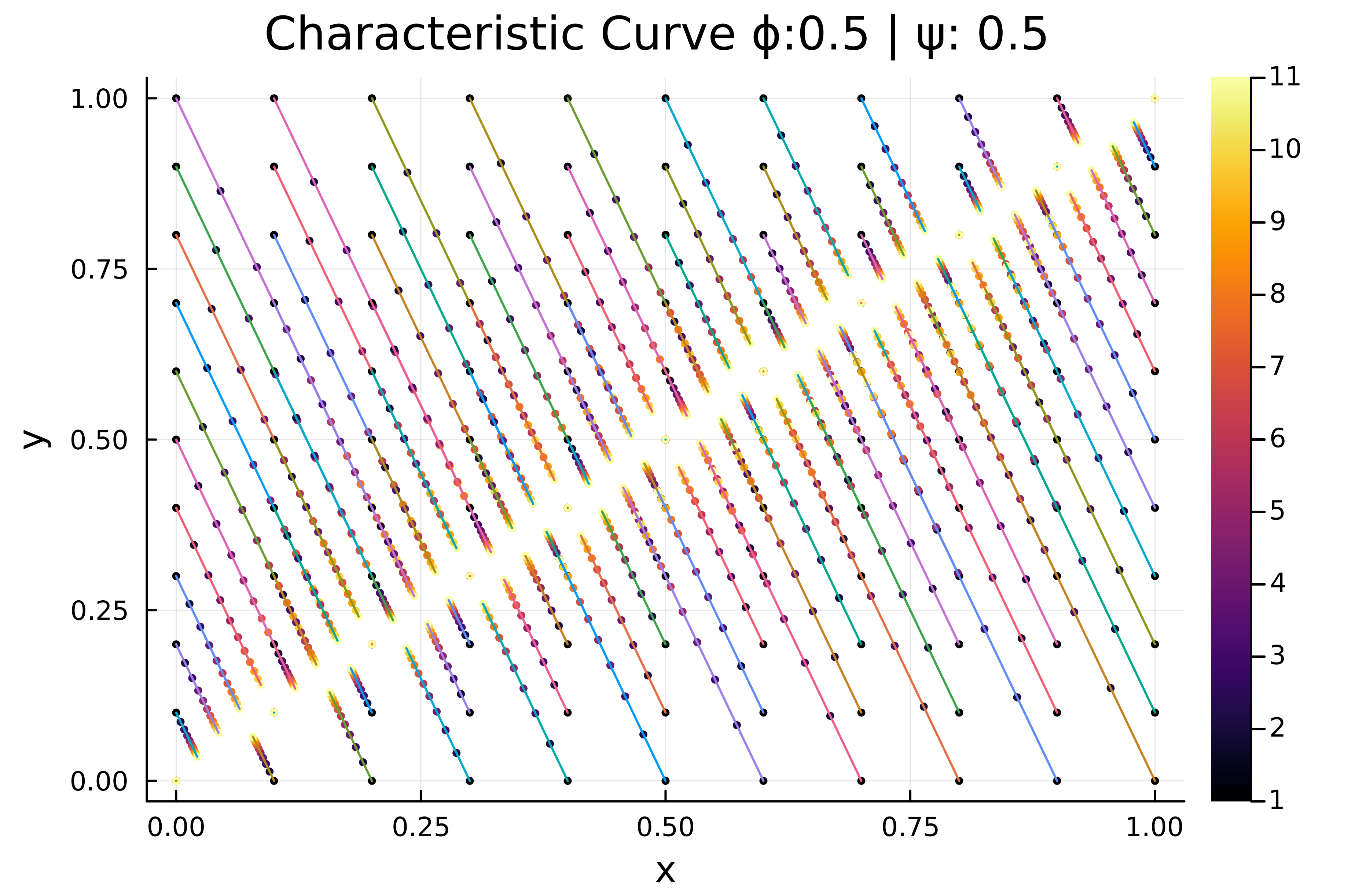
$$e^A := \sum_{k=0}^{\infty} \frac{1}{k!} X^k$$

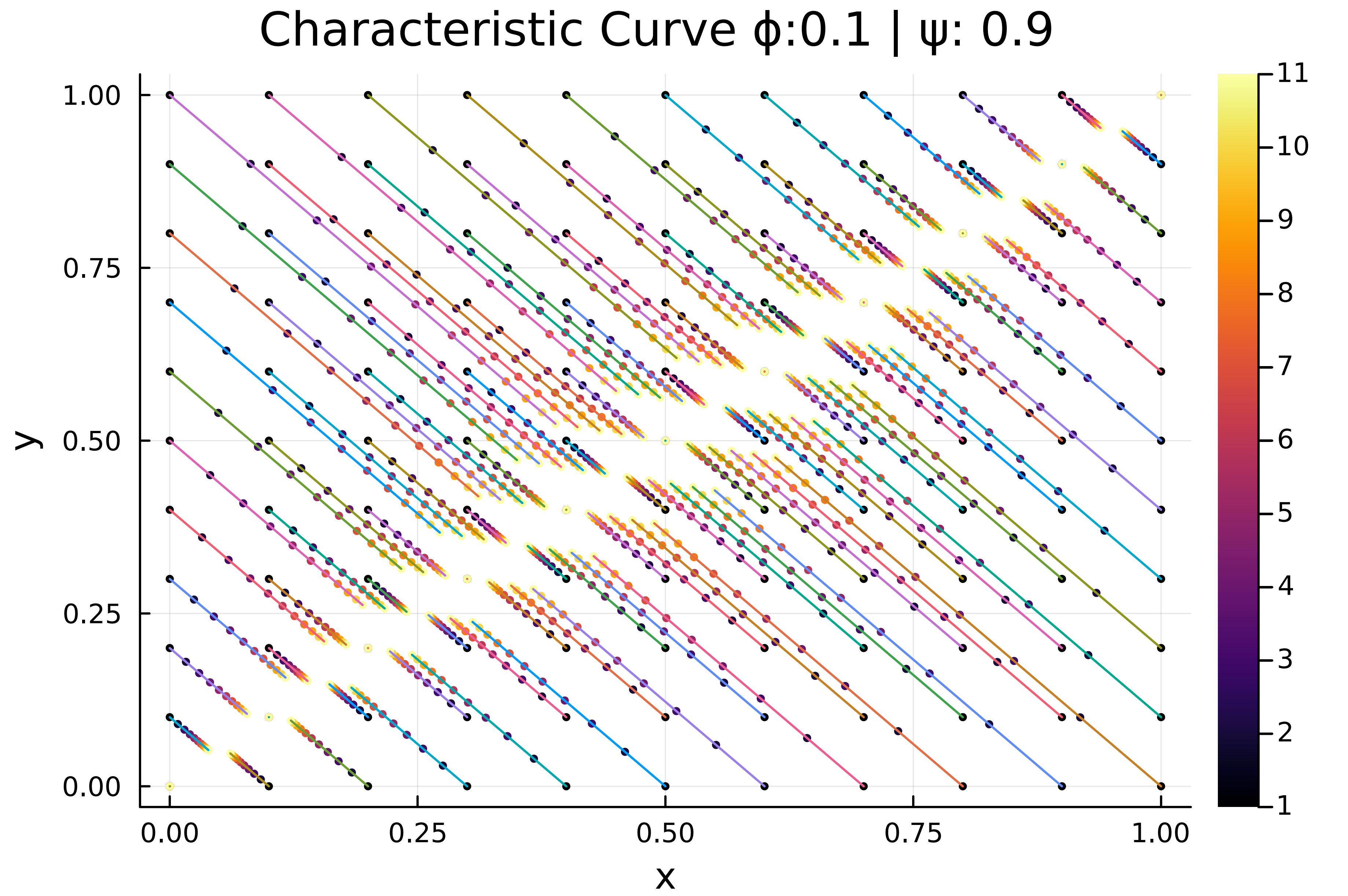
Using the matrix exponential

$$\frac{dt}{ds} = 1 \implies t(s) = s + t_0$$

$$\begin{pmatrix} t(s) \\ x(s) \\ y(s) \end{pmatrix} = \begin{pmatrix} s + t_0 \\ (exp(-As)\mathbf{x_0})_x \\ (exp(-As)\mathbf{x_0})_y \end{pmatrix} \Rightarrow \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} (exp(-At)\mathbf{x_0})_x \\ (exp(-At)\mathbf{x_0})_y \end{pmatrix}$$

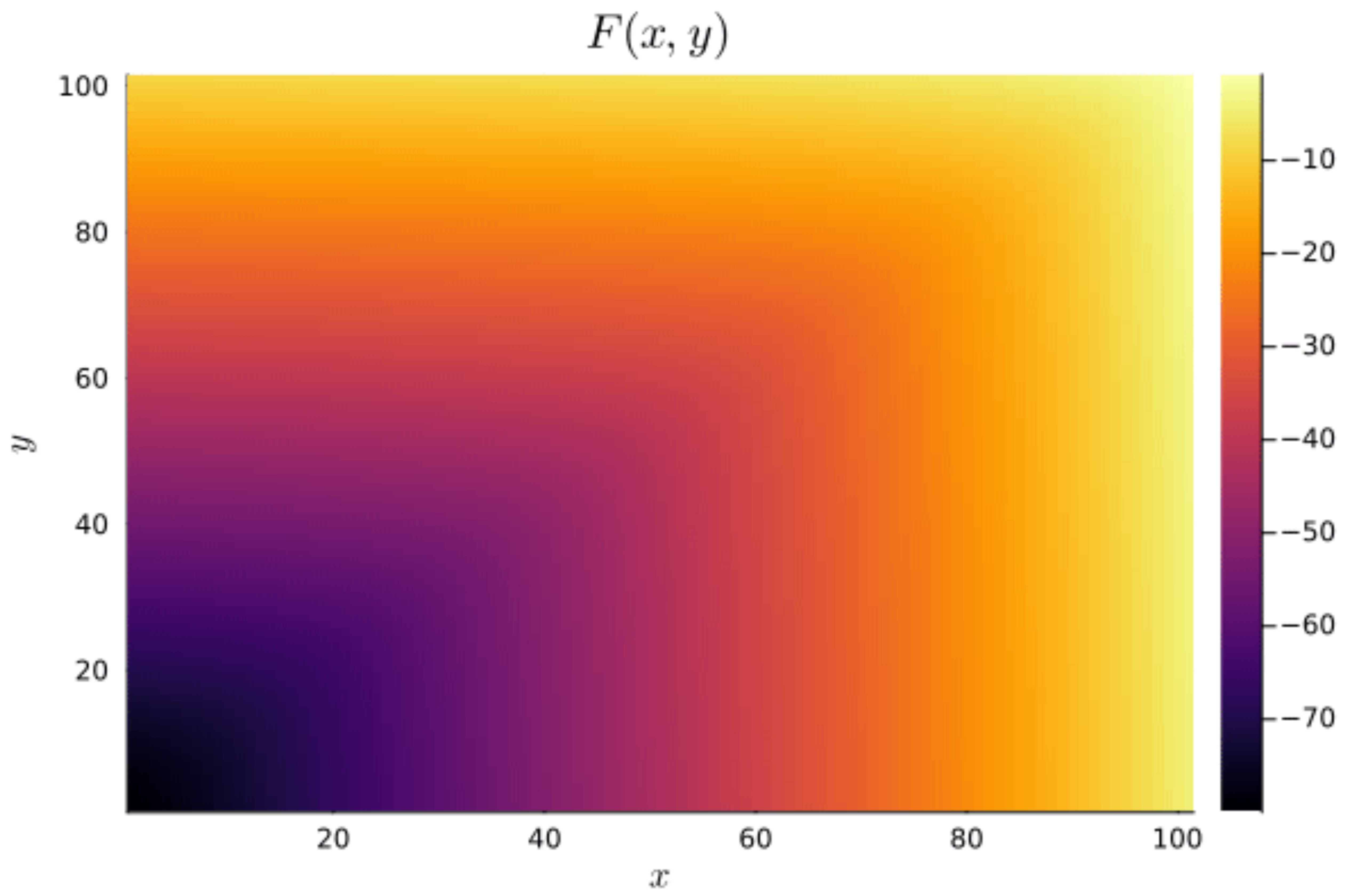


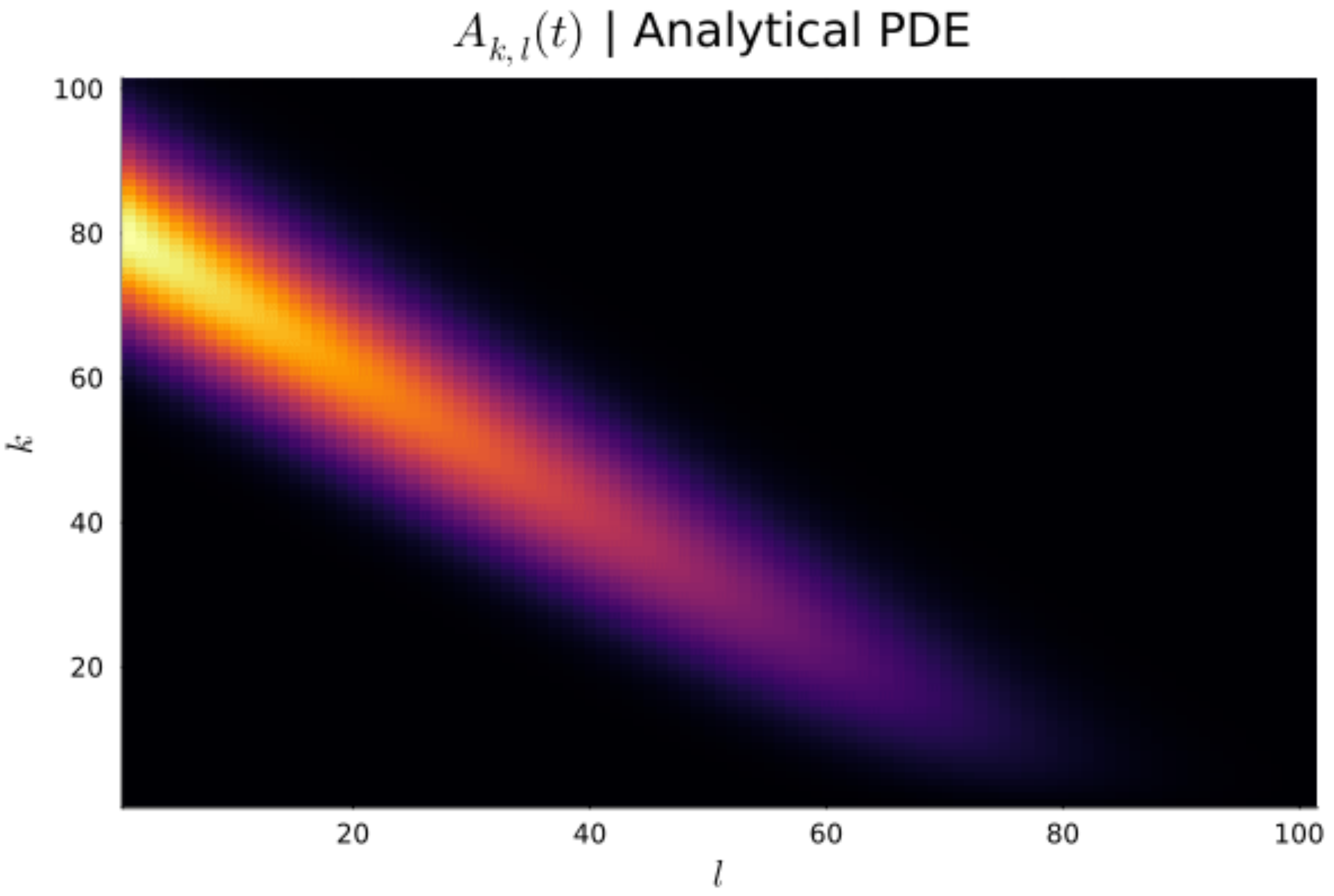




Along characteristics curves $F(x, y, t) = F_0(x_0, y_0)$

$$F_0(x_0, y_0, 0) = F_0((exp(-At)\mathbf{x_0})_x, (exp(-At, t)\mathbf{x_0})_y)$$





Analytical PDE Solution matches numerical ODE solution!

Marginal Distribution of A nodes A_k(numerical)
A_k
0.00
0.01
0.02
0.03
0.04
0
25
50
75
100
k
Marginal Distribution of A nodes A_k(analytical)
A_k
0.00
0.01
0.02
0.03
0.04
0
25
50
75
100
k

Further Work

Examine the effect of changing network topology and degree distribution the steady state

Validate against agent based model